Using Coalition and Network Theory

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Coalition Theory

A simple formalization is as follows:

Agents

$i \in (1, 2, \ldots, I) \equiv \mathcal{I}$

Coalitions:

$C \subseteq \mathcal{I}$

The set of all possible coalitions: $\mathcal{C} \equiv \{C \subseteq \mathcal{I}\}$

Coalitions are collections of agents, formally, subsets of $\mathcal{I}$, which is the set of all agents called the *grand coalition*.

Allocations:

X^C is called an allocation for the coalition C. This is a list payoffs $(x_i^C, x_j^C, x_k^C, \ldots) \equiv X^C$ where $i, j, k, \in C$.

We will denote an allocation for the grand coalition as $X^{\mathcal{I}}$

One of the simplest ways to describe what is feasible for a coalitions is to use a Characteristic or Value Function.

$$V : \mathcal{C} \to \Re$$

This is a Transferable Utility form or a TU function. Value functions can also be Non-Transferable and take an NTU form.

$$V : \mathcal{C} \to \Re^{|C|}$$

From here is is easy to define such things as the **Shapley Value**:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n-|S|-1)!}{n!}(v(S \cup \{i\}) - v(S))$$

Or the core:

An allocation for the grand coalition $x^{\mathcal{I}}$ is ***blocked*** by an allocation x^c for coalition C if

- $\sum_{i \in C} x_i^C \leq V(C)$ (x^C is feasible for C)
- For all $i \in C, x_i^c \geq x_i^{\mathcal{I}}$ (all agents in the blocking coalition are at least as well off)
- For some $j \in C, x_j^c \geq x_j^{\mathcal{I}}$ (some agents in the coalition C are strictly better off)

Core: $x^{\mathcal{I}}$ is a core allocation if it cannot be blocked.

A special case is an anonymous value function:

$$V : N \to \mathfrak{R}$$

Which maps the natural numbers into transferable payoff. Here, only the number of agents in the coalition matter, not the specific agents.

We can say that an anonymous games $(\mathcal{I}, V)$ is **super-additive** if for all $C, C' \subseteq \mathcal{I}, s.t.\ C \subseteq C'$ $\frac{V(C)}{|C|} \leq \frac{V(C')}{|C'|}$.

This says that the per capita payoff increases with coalition size.

In general, if a game (perhaps induced by an economy) is supper-additive, then the core will be non-empty.

If it the game strongly supper-additive, the core will be large.

Otherwise, the core may be empty.

Examples:

Public goods with exclusion: large core

Public goods without exclusion: empty core

Private goods: core exists, and core convergence. Why?

Consider the following example: Suppose there were six agents in the economy with the following anonymous TU value function:

$V(1) = 1$
$V(2) = 5$
$V(n) = 2n$ for $n > 2$

What allocations are in the core?

None! The game is not super-additive.

However, why do we insist that agents live in the grand coalition in this case?

Instead they could break into two person coalitions with each agent getting a payoff of 2.5.

Is this in the core? Yes.

We can think of this as a model of marriage.

One is lonely.

Two is much better

Three lowers per capita payoff.

This suggests two ways to advance:

- Local public goods and club goods
- Matching models

The idea of LPG's is from Tiebout (1956) who noted that fundamental problem with public goods provision is that we get free riding.

This is because:

- Exclusion is sometimes quite hard to implement.
- Agents have an interest in hiding their willingness to pay for public goods.

Tiebout noticed, however, that many public goods are not provided by the central government, but by states and localities instead.

Examples:

- Schools
- Sewers
- Police
- Fire
- Parks
- Universities
- Roads

Localities offer offer bundles of public amenities and tax prices. Agents would look across the menu of these offers and “vote with their feet”,

By choosing the best offer, they reveal their willingness to pay.

More formally, public goods would be provided by coalitions that form an exhaustive partition of all the agents.

The the generalized value function approach looks at coalitional formation games as abstracted from the underlying economy or strategic environment. It is a sort of reduced form. LPG instead exploits the details of the economy to consider how coalitions are formed and what they do.

The fundamental element that turns a pure public good into a LPG is "crowding".

What is this?

Here is a very simple model of anonymous crowding:

Consider following model:

One private good x (more are easily added)
One public good y (more are easily added)

Agents:

$i = 1, \ldots I$

There are a total of T possible partisans of taste, and each of the agents has one of these.

$t=1,\ldots,T$

There are N_t agents of type t in the economy. Thus, the population can be described as

$(N_1, \ldots, N_t, \ldots, N_T) = N$

Agents form jurisdictions that make up a *partition* of the population. That is, each agent is in one and only one jurisdiction.

A given jurisdiction is described as

n^j = the number of agents in jurisdiction j.

The number of agents in each jurisdictions j is given by the list:

$n = (n^1, ..., n^j, ..., n^J)$

Agents care about three things:

- public goods in the jurisdiction
- private goods
- the number of agents in the jurisdiction

$$U_t(x, y^j, n^j)$$

Production:

$$F(y, n)$$

gives cost in terms of private good of producing y public good with n agents in jurisdictions. Thus, crowding can also take place in production.

Examples:

- Swimming pools
- Standing in line
- People in your class
- People at a concert
- Cars on the road.

Note that since crowding is anonymous, we only need to know how many agents are in each jurisdiction, not the specific agents who are there.

Let's look back the value function example above. We agreed that three coalitions with two agents each getting 2.5 was a core allocation. In fact it is the only one.

This displays the **equal treatment property** which says that identical agents must get identical allocation in any core state.

What is driving this is that the game satisfies **small groups effectiveness** which requires that all per capital gains can be realized by groups of agents that are small relative to the whole population.

Another property of the core allocation if the economy satisfies SGE is that it is **taste-homogeneous**. That is, all coalitions will have only one type of agent. Why?

Yet another property of LPG economies satisfying SGE is that the **core and Tiebout equilibrium are equivalent**. The core is decentralized in this case by "admission-tax/prices" the give the cost of joining any type of jurisdiction (any level of public goods, and number of agents). This is because SGE implies a kind of linear additivity in coalition size.

Prices are anonymous and take the form:

$p(n^j, \; y^j)$

Finally, again consider the value function example. What if we had five agents? Then the core would be empty. The same is true or LPG economies. Unless the number of agents divides evenly into optimally sized, per capital utility maximizing jurisdictions, there will be left-over agents and the core will be empty.

However, as the economy gets large the ratio of these left-overs to the size of the population goes to zero, and so the approximate core and equilibrium exist. The exact core and equilibrium also exist in continuum versions of LPG economies.

Crowding types:

So far we have only looked at anonymous crowding. All agents are identical to one another.

What if agents crowded each other differently. For example, Girls and Boys are not identical and don't have identical external effects.

Formally:

Each agent now has two characterizes, taste type and crowding type. As before, the utility function is indexed by t:

$t=1,\ldots,T$

Agents also have a crowding type indexed by c:

$c= 1, \ldots, C$

Examples:

- Charisma
- Height
- Gender
- Abilities

In general, these are payoff relevant aspects of agents that are exogenous to agents. Crowding effects could be positive or negative. We care about how many of each type are in our coalition.

There are N_{ct} agents of type ct in the economy. Thus, the population can be described as

$$(N_{11}, \ldots, N_{ct}, \ldots, N_{CT}) = N$$

A given jurisdiction is described as:

$n^j = (n^j_1, \ldots, n^j_c, \ldots, n^j_c)$

$n^j_c =$ the number of agents of crowding type c in jurisdiction j.

Agents care about three things:

- public goods in the jurisdiction
- private goods
- the number of each crowding type of agents in the jurisdiction

$$U_t(x, y^j, n^j_1, \ldots, n^j_c, \ldots n^j_C).$$

Production:

$$F(y, n_1, \ldots, n_c, \ldots n_C)$$

For crowding types, we get:

- Equal treatment
- No homogeneity

<u>Endogenous crowding type</u>

One thing that is assumed in the above discussion is that crowding is exogenous. However, many things that affect others are chosen. To deal with this simply add and "educational cost functions"

$$E(c) = x$$

Prices continue to take the form:

$$p_c(n^j_1, \ldots, n^j_C, \; y^j)$$

Here we get

- Equal treatment
- Homogeneity

<u>Endogenous crowding type with genetic differences.</u>

This model may be too simple as it assumes that agents are equally able to acquire different crowding characteristics. However, in real life, people have different types of innate abilities. These abilities themselves do not generate crowding externalities, but they affect the cost of acquiring different abilities. Agents now have two basic characteristics, tastes and genetic types:

$t = 1, \ldots, T$
$g = 1, \ldots, G$

Examples of genetic types:

- Intelligence
- Athletic ability
- Artistic talent
- Memory

The educational cost function now depends on this:

$$E(c, g) = x$$

We can also imagine that people care about the crowding type they express. Formally, this means that we must extend the utility function:

$$U_t(x, y^j, c, n^j{}_1, \ldots, n^j{}_c, \ldots n^j{}_C).$$

Prices continue to take the form:

$$p_c(n^j_1, \ldots, n^j_C, \; y^j)$$

Here we get

- Equal treatment
- No homogeneity

<u>Variable usage</u>

So far we have only made membership in a club relevant. In real life, we may care about the extent of participation. For example, we might join a county club first of all for the membership (crowding profile). After all, being a member of the right club pays dividends. However, a secondary reason might be to play golf.

Here we are crowded negatively by the total number of visits that other agents make to the course (V).

We also care about how many visits we make to the course (v^i).

Thus:

$$U_t(x, y^j, n^j_1, \ldots, n^j_c, \ldots n^j_C, v^i, V).$$

Prices are now two-part. First there is the fixed joining price that depends on the qualities of the club:

$$p_c(n^j_1, \ldots, n^j_C, y^j, V)$$

There is also a per visit price (greens fees):

$$q_c(n^j_1, \ldots, n^j_C, y^j, V)$$

This models:

- Country clubs
- Places of work
- Roads and Bridges

One could take any one of these models of coalition formation with crowding and add variable usage.

Multiple membership clubs

What about multiple membership clubs?

I might join a school, a church, a workplace and a family. Existence is problematic in the same way as simple membership clubs.

This also leads a question of commitment to clubs.

In all of these cases one has to determine the rules of coalition formation.

Price taking: This implies the presence of some kind of firm, mayor, or developer, and also free mobility/free entry. Often, equilibrium does not exist.

Nash: Unilateral action only is allowed, but with free mobility/entry. In general, Nash equilibrium of LPG games will exist but not be PO.

Core: Allows coalitional actions and also allows exclusion. In general, the core is PO but may not exist

Other rules:

- More complicated mechanisms
- Refinements of Nash
- Generalizations of the core

Matching problems

Matching problems are a special case of coalition formation.

- Marriage problems
- Two sided markets
- Kidney donor and recipient

Sometimes matching is one to many:

- Interns to residencies
- Students of schools
- Workers to firms

or many to many

- Friends to friends
- Multi-unit markets

Many of these problems focus on the value of the other agents, either his or her characteristics, or the value of their offer compared others (markets).

In some cases, it is the amenity that is valued, for example, the house, the kidney, the reputation of the school, the wage offered by the firm. This is similar to LPG.

In one to one matching especially, assortative matching is often an equilibrium.

The matching literature has especially focused on:

- **Mechanism design:** Finding way to produce matches are optimal or at least stable.

- Institutions: looking at models of existing real word institutions to see how people match. For example, the Rubinstein-Wolinsky bargaining game, and auctions

- **Incomplete information and search theory** are more central here then in LPG or coalition theory generally.

Networks

Coalition theory is very flexible and can be used to model many of our economic and social interactions: where we live, who we marry, where we work etc.

It can easily describe how we affect one another in groups depending on our characteristics and consumptions choices.

However, embedded in coalitional approaches is the idea that all members of a coalition are equally in contact.

Membership in binary: you are in or you are out of a coalition.

Networks are a different way of modeling the relationships between agents.

The idea is that individual pairs of agents form links between one another. Think of these as friendship links, for example.

Formally:

Agents

$i \in (1,2,\ldots,I) \equiv \mathcal{I}$

An undirected link between two players is denoted by an unordered pair $\{i,j\}$.

An undirected network is denoted g and is a collection of such links.

More precisely let g^N denote the set all two element subsets of $\mathcal{I}$. Then the set of all possible networks on $\mathcal{I}$ is denoted $G \equiv \{g \subseteq g^N\}$.

Note the g^N itself is a special case of a network called the complete network.

Alternatively, the ordering of the pair that forms the link could be material. If we take $\{i,j\}$ as an ordered pair, it describes a directed link from i to j in a directed network. A directed link might only allow traffic or information to flow in that direction, or it could indicate that i has made offer of friendship to j.

Value functions are also useful in network theory. Here $V:G\rightarrow\Re$. Of course we also have the analogue of NTU and anonymous value functions as well.

Given this, $\{N,V\}$ defines a (cooperative) network game.

Let $\mathcal{V}$ denote the set of all possible value functions. An allocation rule is a mapping $Y:G\times\mathcal{V}\rightarrow\Re^N$ such that $\sum_i Y_i(g,v)=v(g)$ for all v and g.

One literature takes network theory in the allocation theory direction. It suggests the sharing of network value according to axioms of fairness based on the network structure.

Another literature goes in the strategic direction and considers different rules of network formation.

The only literature I am aware of that uses price taking in networks focuses on allocating scare network capacity to users, and this is mostly a computer science literature.

Links have a wide set of possible interpretations.

- Friendship
- Communication links
- Trade links
- Coauthorship
- Professional link

In some cases, networks are exogenously given and agents behave strategically within the constraints this imposes.

In others, the networks are endogenously formed. In this case, we have to know what the rules of network formation are.

- Are links costly to build?
- Are both parties forced to share the costs, or just the party imitating the link?
- What order to agents move in, or are moves simultaneously?
- Are links directed or undirected?
- What is the equilibrium notion?
    - Nash
    - SGP
    - Sequential
    - Pairwise stable
    - Farsightedly stable.

If a group of agents is totally connected by undirected links, then this is really the same as being in a coalition.

Even in the case where agents are not totally connected, we can describe all first order links as an overlapping club structure. That is, every subgroup that is totally connected is a club. In some cases, clubs will consist of only two players, of course.

However, it may be that I have to go through one or more people (nodes) to communicate with another agent in my network. This may be costly, less likely to occur, or it may be that such extended connections confer fewer external benefits.

Thus, for networks to be better than coalitions as a modeling approach:

- there must be something about the number of links it takes to reach another agent that is both driven by the economic phenomena you wish to model and also provides some economics insight.
- the rules of the network formation game must be economically motivated and not-well treated as a coalition formation game.

There are other features that we could add to a network model that would expand it application.

Links might have capacity. In neural nets, the strength of a link between neurons depends on how often the pathway is used. Friendship networks might have the same property.

Nodes need not be people, nor links represent some type of social connection. It might be that the nodes are DNS servers and the links, Internet backbone. Or perhaps the nodes are cites and power generation sites, and links, rails, roads and power-lines.

Road networks, water distribution networks, power networks, communications networks require the creation of costly links between nodes that are not agents.

The network structure is usually chosen by an agent who is trying to minimize or maximize some objective like least cost, or least latency.

Agents within the network might be passive (water), active optimizers (agents choosing which route to take to work) or even artificial (DNS servers choosing a route for packets based on current network conditions).

Nodes may also have capacity or be otherwise differentiated. Certainly, one can imagine that DNS server nodes have a fixed capacity to pass through packets from incoming to outgoing links. In networks of people, different agents might have different values. This is similar to crowding, but now the effects depends on the network connections, not coalition membership.

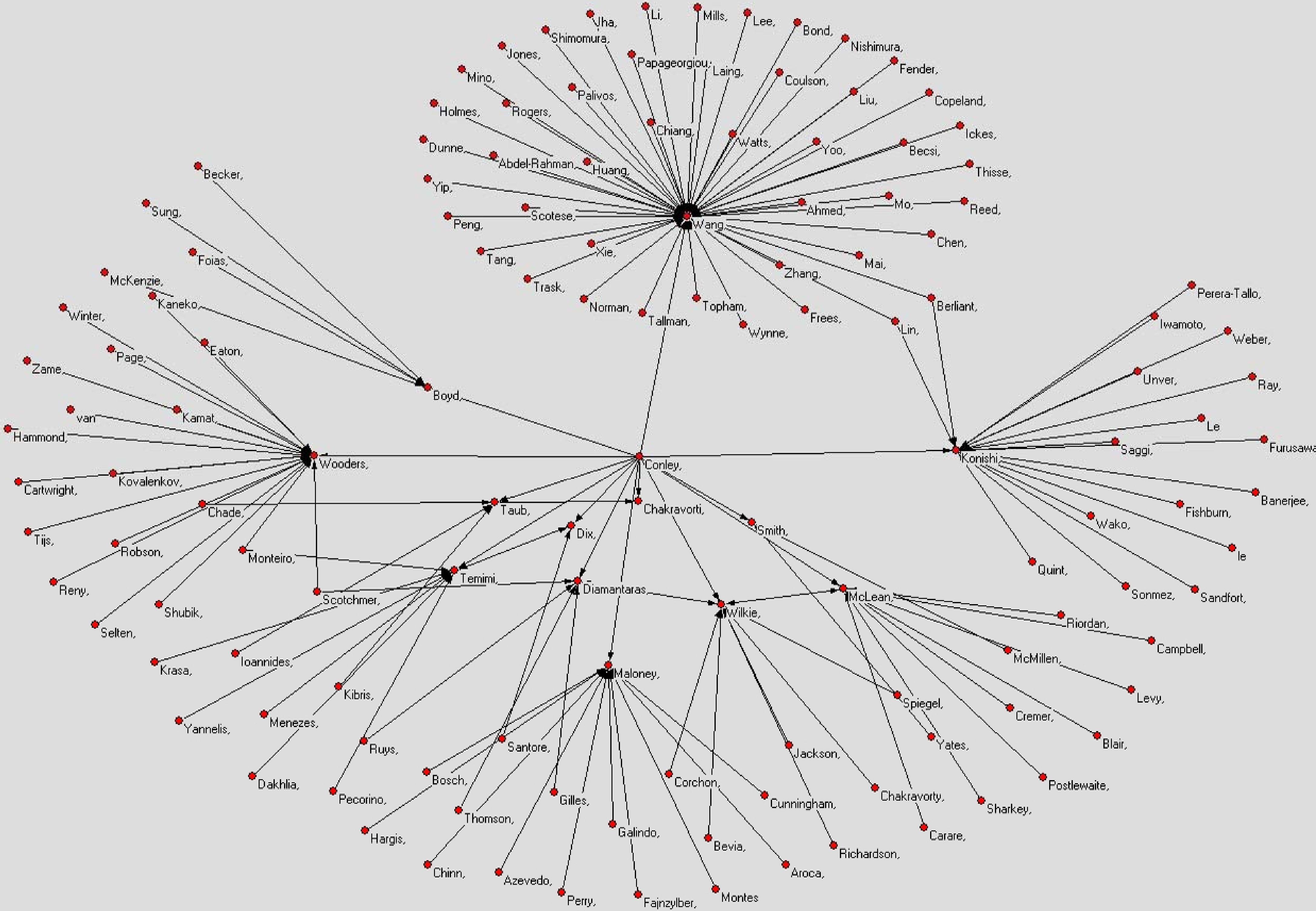
Consider coauthor networks for example, Everyone I write with is linked to me. Everyone my coauthors write with is linked to me in two steps, and so on. The prestige of my network may affect my own prestige or productivity. Being linked in friendship networks to the cool kids may yield similar benefits.

So we see that networks model many of the same things that coalitions do:

- Positive and negative crowding
- Crowding by types
- Partial commitment/variable usage
- Free mobility (can't refuse to take an incoming link) or exclusivity (the opposite)

However, they add a structure of connections between agents or nodes that define connection more finely than “in or out”.

Like always, the trade off is between generality and results. Networks allow for deeper descriptions of connections between agents. On the other hand, the richness that networks allow makes it harder to prove specific results.



Jha,
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Spiegel,
Cremer,
Yannelis,
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Jackson,
Postlewaite,
Dakhlia,
Bosch,
Corchon,
Chakravorty,
Sharkey,
Cunningham,
Pecorino,
Gilles,
Thomson,
Carare,
Hargis,
Galindo,
Bevia,
Richardson,
Chinn,
Azevedo,
Aroca,
Perry,
Fajnzylber,
Montes

Table 1. Number of AER-Equivalent Publications of Graduating Classes from 1986 to 2000

Department	*Percentiles of Graduates' AER-Equivalent Publications 6 years after Ph.D.*												
	99th	95th	90th	85th	80th	75th	70th	65th	60th	55th	50th	45th	40th
Harvard	4.31	2.36	1.47	1.04	0.71	0.41	0.30	0.21	0.12	0.07	0.04	0.02	0.01
Chicago	2.88	1.71	1.04	0.72	0.51	0.33	0.19	0.10	0.06	0.03	0.01	0.01	0
U Penn	3.17	1.52	1.01	0.60	0.40	0.27	0.22	0.13	0.06	0.03	0.02	0.01	0
Stanford	3.43	1.58	1.02	0.67	0.50	0.33	0.23	0.14	0.08	0.05	0.03	0.02	0.01
MIT	4.73	2.87	1.66	1.24	0.83	0.64	0.48	0.33	0.20	0.12	0.07	0.04	0.02
UC Berkeley	2.37	1.08	0.55	0.35	0.20	0.13	0.08	0.06	0.04	0.03	0.02	0.01	0.01
Northwestern	2.96	1.92	1.15	0.93	0.61	0.47	0.30	0.21	0.14	0.10	0.06	0.03	0.01
Yale	3.78	2.15	1.22	0.83	0.57	0.39	0.19	0.12	0.08	0.05	0.03	0.02	0.01
UM Ann Arbor	1.85	0.77	0.48	0.29	0.17	0.09	0.05	0.03	0.02	0.01	0.01	0	0
Columbia	2.90	1.15	0.62	0.34	0.17	0.10	0.06	0.02	0.01	0.01	0.01	0	0
Princeton	4.10	2.17	1.79	1.23	1.01	0.82	0.60	0.45	0.36	0.28	0.19	0.12	0.09
UCLA	2.59	0.89	0.49	0.26	0.14	0.06	0.04	0.02	0.02	0.01	0	0	0
NYU	2.05	0.89	0.34	0.20	0.07	0.03	0.02	0.01	0.01	0.01	0	0	0
Cornell	1.74	0.65	0.40	0.23	0.12	0.07	0.05	0.04	0.02	0.01	0.01	0.01	0
UW Madison	2.39	0.89	0.51	0.31	0.20	0.11	0.06	0.04	0.03	0.02	0.01	0.01	0
Duke	1.37	1.03	0.59	0.49	0.23	0.19	0.11	0.08	0.05	0.04	0.02	0.01	0
Ohio State	0.69	0.41	0.13	0.07	0.04	0.02	0.02	0.01	0.01	0.01	0	0	0
Maryland	1.12	0.37	0.23	0.10	0.07	0.05	0.03	0.02	0.01	0.01	0.01	0	0
Rochester	2.93	1.94	1.56	1.21	1.14	0.98	0.70	0.51	0.34	0.27	0.17	0.12	0.06
UT Austin	0.92	0.53	0.21	0.06	0.05	0.02	0.01	0.01	0	0	0	0	0
Minnesota	2.76	1.20	0.68	0.46	0.29	0.21	0.12	0.08	0.04	0.02	0.01	0.01	0
UIUC	1.00	0.38	0.21	0.10	0.06	0.04	0.03	0.02	0.01	0.01	0.01	0	0
UC Davis	1.90	0.66	0.42	0.27	0.12	0.08	0.05	0.03	0.02	0.02	0.01	0	0
Toronto	3.13	1.85	0.80	0.61	0.29	0.19	0.15	0.10	0.07	0.05	0.03	0.02	0.02
UBC	1.51	1.05	0.71	0.60	0.52	0.45	0.26	0.23	0.22	0.15	0.11	0.08	0.05
UCSD	2.29	1.69	1.17	0.88	0.74	0.60	0.46	0.34	0.30	0.20	0.18	0.10	0.06
USC	3.44	0.34	0.14	0.09	0.03	0.02	0.02	0.01	0.01	0	0	0	0
Boston U	1.59	0.49	0.21	0.08	0.05	0.02	0.02	0.01	0	0	0	0	0
Penn State	0.93	0.59	0.25	0.12	0.08	0.06	0.02	0.02	0.01	0.01	0.01	0	0
CMU	2.50	1.27	1.00	0.86	0.71	0.57	0.52	0.29	0.21	0.13	0.09	0.08	0.05
Non-top 30	1.05	0.31	0.12	0.06	0.04	0.02	0.01	0.01	0	0	0	0	0

Table 2: The Number of Graduates each year for each Department who Publish at Least a Given Number of AER Equivalent Papers within 6 years										
AER Papers	2.5	2	1.5	1.25	1	0.75	0.5	0.25	0.1	**Av. Cohort Size**
Harvard	1.3	2.1	2.9	3.9	4.6	5.8	7.2	10.1	12.7	30.5
Chicago	0.5	0.9	1.7	2.1	3.1	4.0	5.6	7.5	9.5	27.3
U Penn	0.4	0.7	1.1	1.3	1.9	2.3	3.5	5.5	7.1	19.3
Stanford	0.7	0.9	1.4	1.7	2.7	3.4	5.0	7.4	9.3	24.7
MIT	1.5	2.0	3.1	3.8	4.7	5.4	7.5	9.9	11.9	25.5
Berkeley	0.3	0.5	0.9	1.1	1.8	2.1	3.1	5.2	7.9	28.0
Northwestern	0.3	0.5	0.8	0.9	1.3	2.0	2.5	3.3	4.5	10.1
Yale	0.7	0.9	1.3	1.5	1.9	2.5	3.5	4.5	5.9	15.7
UM Ann Arbor	0.0	0.1	0.4	0.5	0.7	1.0	1.8	3.3	4.7	19.1
Columbia	0.3	0.3	0.5	0.7	1.1	1.6	2.3	3.1	4.3	17.4
Princeton	0.7	1.2	2.0	2.3	3.3	4.4	5.4	7.6	9.4	16.2
UCLA	0.2	0.2	0.5	0.5	0.8	1.1	1.7	2.7	3.9	17.9
NYU	0.0	0.1	0.1	0.3	0.4	0.6	1.0	1.6	2.1	11.7
Cornell	0.1	0.1	0.3	0.3	0.4	0.7	1.3	2.4	3.8	17.3
UW Madison	0.0	0.3	0.5	0.6	1.1	1.7	2.6	4.3	6.4	25.0
Duke	0.0	0.0	0.0	0.2	0.4	0.6	1.1	1.5	2.4	7.8
Ohio State	0.0	0.0	0.0	0.1	0.1	0.1	0.5	1.1	1.7	15.9
Maryland	0.0	0.1	0.1	0.1	0.3	0.3	0.4	1.3	2.2	13.5
Rochester	0.1	0.3	1.0	1.2	2.1	2.5	3.1	4.1	4.9	8.7
UT Austin	0.0	0.0	0.0	0.0	0.1	0.2	0.6	0.9	1.4	10.3
Minnesota	0.4	0.6	0.8	1.1	1.4	1.9	2.9	4.8	7.1	22.2
UIUC	0.0	0.0	0.1	0.2	0.3	0.4	1.1	2.2	3.9	26.4
UC Davis	0.0	0.0	0.1	0.1	0.1	0.2	0.5	1.0	1.3	6.2
Toronto	0.1	0.2	0.3	0.5	0.5	0.7	1.1	1.5	2.3	6.4
UBC	0.0	0.0	0.1	0.1	0.3	0.4	0.9	1.5	2.3	4.5
UCSD	0.0	0.1	0.5	0.6	0.7	1.2	1.8	2.5	3.4	6.1
USC	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.4	0.7	4.9
Boston U	0.0	0.1	0.1	0.1	0.2	0.3	0.5	1.1	1.8	12.5
Penn State U	0.0	0.0	0.0	0.1	0.1	0.3	0.5	0.8	1.2	7.1
CMU	0.0	0.1	0.1	0.1	0.2	0.4	0.6	0.8	0.9	2.0
Non-top 30	0.0	0.0	0.1	0.1	0.2	0.3	0.5	1.0	1.8	16.8

Table 3. Department Rankings based on Graduating Cohorts' Publication Performance (1986-2000)														
Department	Coupe	Ranking at Percentile:												
Percentile		99th	95th	90th	85th	80th	75th	70th	65th	60th	55th	50th	45th	40th
Harvard	1	2	2	4	4	5	8	6	8	8	8	8	11	11
Chicago	2	12	8	8	9	10	10	12	13	12	15	17	12	30
U Penn	3	7	11	10	13	12	12	10	10	13	13	14	15	14
Stanford	4	6	10	9	10	11	11	9	9	9	9	10	9	10
MIT	5	1	1	2	1	3	3	4	4	6	6	6	6	6
UC Berkeley	6	17	15	17	16	17	16	16	16	15	14	13	14	12
Northwestern	7	9	6	7	5	7	6	7	7	7	7	7	7	9
Yale	8	4	4	5	8	8	9	11	11	10	11	11	10	8
UM Ann Arbor	9	21	21	20	19	18	19	21	20	20	20	23	21	23
Columbia	10	11	14	15	17	19	18	18	21	22	23	20	30	21
Princeton	11	3	3	1	2	2	2	2	2	1	1	1	2	1
UCLA	12	14	19	19	21	20	22	22	22	21	22	26	26	17
NYU	13	19	20	23	23	24	26	26	27	27	27	30	27	22
Cornell	14	22	23	22	22	21	21	19	18	19	19	15	18	18
UW Madison	15	16	18	18	18	16	17	17	17	17	17	19	16	13
Duke	16	25	17	16	14	15	15	15	15	14	12	12	13	19
Ohio State	17	31	27	30	29	29	27	27	26	24	26	28	24	25
U Maryland	18	26	29	25	25	25	24	23	25	25	21	21	19	27
Rochester	19	10	5	3	3	1	1	1	1	2	2	3	1	2
UT Austin	20	30	25	27	31	27	29	31	31	31	28	27	25	20
Minnesota	21	13	13	14	15	14	13	14	14	16	16	18	17	26
UIUC	22	28	28	26	26	26	25	24	24	26	25	24	28	31
UC Davis	23	20	22	21	20	22	20	20	19	18	18	16	20	28
Toronto	24	8	7	12	11	13	14	13	12	11	10	9	8	7
UBC	25	24	16	13	12	9	7	8	6	4	4	4	4	5
UCSD	26	18	9	6	6	4	4	5	3	3	3	2	3	3
USC	27	5	30	29	27	31	28	28	28	28	30	25	31	15
Boston U	28	23	26	28	28	28	30	29	29	30	31	29	22	24
Penn State	29	29	24	24	24	23	23	25	23	23	24	22	29	16
CMU	30	15	12	11	7	6	5	3	5	5	5	5	5	4
Non-top 30		27	31	31	30	30	31	30	30	29	29	31	23	29