Convergence Theorems on the Core of a Public Goods Economy†

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Abstract

In 1963, Debreu and Scarf were at last able to prove Edgeworth's conjecture that the core of an exchange economy converges to the competitive equilibria as the number of traders gets large. Unfortunately, Muench(1973) demonstrated that this result can not be generalized to a public goods economy This is due to the increasing returns to coalitional size that are embedded in the public goods technology. The purpose of this paper is to investigate the conditions under which core convergence can be regained when public goods are present. It is shown that if all consumers become asymptotically satiated in public good, then the core of a replication economy converges to the set Lindahl equilibria. *Asymptotic satiation* means that the value to consumers, in terms of private goods, of receiving some multiple of their current level of public goods consumption goes to zero as the level goes to infinity. This assumption is trivially implied if consumers become satiated in public good. It is also shown that strict non-satiation, which is almost the inverse of asymptotic satiation, is a sufficient condition for core convergence. *Strict non-satiation* mean that consumers' marginal rates of substitution of public for private goods is strictly bounded from zero. Finally sufficient conditions for core non-convergence are investigated.

1. Introduction

In 1963, Debreu and Scarf proved Edgeworth's conjecture that the core of an exchange economy converges to the set of competitive equilibrium allocations as the number of traders gets large. It would be quite surprising, of course, to observe an allocation that is not in the core as an equilibrium outcome of any game. This includes the one induced by an exchange economy, or any other economy. Such an outcome would violate the basic economic hypothesis that agents always do as well as they can given their constraints. Thus the knowledge that the competitive allocations are in fact the *only* ones that remain in the core as the economy gets large is of great practical importance. It confirms the central position of competitive equilibrium in economic theory, and provides a justification for the study of such subjects as comparative statics, stability, and computable general equilibrium.

It is natural to wonder whether or not this result can be generalized to an economy with public goods. Here the question becomes: Does the core of a public goods economy converge to the set Lindahl equilibrium allocations. [1] Muench (1973) was able to demonstrate that such a generalization is not possible. He provided a simple example with a continuum of identical consumers in which the core contains many allocations while the Lindahl equilibrium is unique. The increasing returns to coalitional size that are embedded in the public goods technology are at the root of his result. In essence, the presence of public goods causes the game generated by the economy to be highly super-additive, which in turn causes the core to be large. This calls into question the value of paying the same careful attention to Lindahl equilibrium that has be given to competitive equilibrium.

The main purpose of this paper is to investigate the conditions under which core convergence can be regained when public goods are present. The question is addressed using a replica, instead of a continuum economy. This is because it is difficult to interpret allocations in a public goods economy with a continuum of consumers. For

[1] The Lindahl equilibrium is the natural analog of competitive equilibria for a public goods economy. See Samuelson (1954), and Foley (1970) among others.

example, if there is *any* measurable level of public goods in an allocation, then the ratio of public to private goods consumption for all consumers is infinity. How does one compare two such allocations. After all, infinity is infinity. Is it meaningful to say that consumers prefer a larger infinity of public goods to a smaller one? On the other hand, it is fundamentally impossible to distinguish between allocations in which the public goods level is unmeasurable (almost zero). For such allocations the ratio of public to private goods consumption for each consumer could be bounded, or undefined. Differentiating these two cases is important, and only possible in a finite economy.

The most important result presented in this paper is that the core of a replica economy with technology that exhibits constant returns to scale, and consumers who have convex, continuous, and monotonic preferences, converges to the set of Lindahl equilibrium allocations if all agents become asymptotically satiated in public goods. An agent's preferences are said to exhibit *asymptotic satiation* if the value to the agent, in terms of private good, of receiving some multiple of his current public goods consumption level goes to zero as the level of public good goes to infinity.

This does not seem to be a terribly unreasonable assumption to make about agents' preferences for many kinds of public goods. For example, if an agent had ten channels on his cable system, he might be willing to pay a lot for ten more. But if he already had one hundred, would he care that much about another hundred? Certainly, if he already had one thousand, adding another thousand would hardly improve his welfare at all. Or take a public park as an example. While a larger park is always better, an agent would probably care less and less about each successive doubling of the park's size. Eventually, the park gets so big that the agent can never hope to see the whole thing in any event. Obviously, asymptotic satiation is trivially satisfied if agents actually become satiated in public goods.

It is also shown that strict non-satiation in public goods is a sufficient condition for core convergence. A consumer's preferences are said to exhibit *strict non-satiation* if the desire for public goods is so strong that the marginal rate of substitution of public for private goods is everywhere strictly bounded from zero. For example, if public and private goods are perfect substitutes, the assumption is satisfied.

Strict non-satiation is a much less realistic assumption than asymptotic satiation. In particular, it implies that almost all the private good is devoted to public goods production for every allocation in the core of a sufficiently large economy. Empirically, this does not seem to be a very common situation. One reason that the implications of this assumption are considered is that it is that strict non-satiation is essentially the polar opposite of asymptotic satiation. It is therefore of some theoretical interest to note that despite this, both assumptions imply core convergence.

A more important reason for its inclusion becomes evident when the problem of sufficient conditions for core non-convergence is taken up in section four. These sufficient conditions turn out to be almost, but not quite, the negation of asymptotic satiation and strict non-satiation. Whether or not the gap between economies for with the core can be shown to converge and not to converge can be completely closed is an open question. While section four shows that the gap is narrow, it falls short of giving necessary and sufficient conditions for core convergence. However, this section does allow the conclusion that there is no hope of proving a general core convergence theorem for a public goods economy. If fact, non-convergence seems to be more common, perhaps even generic, for such economies.

2. The Model

A simple replica economy, $\mathcal{E}^R$, with T types of consumers indexed by $t \in \mathcal{T} \equiv \{1, \dots, T\}$, and R consumers of each type indexed by $r \in \mathcal{R} \equiv \{1, \dots, R\}$, is considered. Each consumer type has complete and transitive preferences $\succeq_t$ defined over $X \equiv \Re_+^{M+1}$, a one private good, M public good consumption set where public goods are indexed by $m \in \mathcal{M} \equiv \{1, \dots, M\}$. A typical consumption bundle in X will be written $(x; y)$ where x is a level of private good, and y is a vector of public goods. Agents will typically be referred to by their number and type. Thus $\{r, t\}$ is agent number r of type t. Superscripts are used to represent agents and subscripts to represent types of public goods. Superscripts and subscripts will also sometimes be used to distinguish different vectors of goods from one another, but this will be done a way that prevents confusion.

The following assumptions are made on $\succeq_t$ for all $t \in \mathcal{T}$.

A1) $\succeq_t$ is continuous.

A2) If $(x; y) \succeq_t (x'; y')$, then for all $\lambda \in [0, 1]$, $\lambda(x, y) + (1 - \lambda)(x'; y') \succeq_t (x'; y')$.
(Weak convexity)

A3) If $(x; y) \geq (x'; y')$, then[2] $(x; y) \succeq_t (x'; y')$; also, if $x > x'$, and $y \geq y'$, then $(x; y) \succ_t (x'; y')$.
(Monotonicity in all goods, and strict monotonicity in the private good)

Define the K dimensional simplex Δ^K as:

$$\Delta^K \equiv \left\{ p \in \Re_+^{K+1} \;\middle|\; \sum_{i=1}^{K+1} p_i = 1 \right\}. \tag{1}$$

Note that monotonicity implies that all prices are non-negative, and in addition, the price of the private good is positive.

Let $Y \subset \Re_- \times \Re_+^M$ denote the production set. A typical feasible production plan is written $(z; y)$ where z is a negative number which is interpreted as the input of private good, and y is a positive output vector of public goods. Define the *marginal cost correspondence for Y, $MC : Y \to \Delta^M$*, as follows:

$$MC(z; y) \equiv \left\{ (p; q) \in \Delta^M \mid (p; q)(z; y) \geq (p; q)(z'; y') \; \forall \; (z'; y') \in Y \right\}. \tag{2}$$

The production set is assumed to satisfy:

B1) Y is a closed convex cone.

B2) $Y \cap \mathbf{R}_+^{M+1} = \{0\}$.

[2] The three kinds of vector inequalities are represented by $\geq, >,$ and $\gg$.

B3) There exists $\phi > 0$ such that for all $m \in \mathcal{M}$, all $(z; y) \in Y$, and all $(p; q) \in MC(z; y)$, $q_m/p \leq \phi$.

Finally, one assumption is made on the ω^t for all $t \in \mathcal{T}$:

C1) $\omega^t > 0$.

Let G^R denote the grand coalition for the economy $\mathcal{E}^R$. A *Lindahl equilibrium* for a coalition $S \subseteq G^R$ is a vector $(x; y; p; q) \in \Re_+^{\mid S \mid} \times \Re_+^M \times \Delta^{\mid S \mid M}$, such that:

$$\forall \, \{r, t\} \in S, \; (p; q^{r,t})(x^{r,t}; y) \leq p\omega^t \qquad (A),$$

$$(x^{r,t}; y) \succeq_t (\hat{x}; \hat{y}) \; \forall \; (\hat{x}; \hat{y}) \in X \text{ s.t. } (p; q^{r,t})(\hat{x}; \hat{y}) \leq p\omega^t. \qquad (B)$$

$$\Big(\sum_{\{r,t\} \in S} (x^{r,t} - \omega^t); y \Big) \in Y. \qquad (C)$$

and

$$\forall \; (\hat{z}; \hat{y}) \in Y,$$

$$\left(p; \sum_{\{r,t\} \in S} q^{r,t} \right) \left(\sum_{\{r,t\} \in S} [x^{r,t} - \omega^t]; y \right) \geq \left(p; \sum_{\{r,t\} \in S} q^{r,t} \right) (\hat{z}; \hat{y}) \, . \qquad (D)$$

Now define the set of Lindahl allocations for a coalition $S \subseteq G^R$ as:

$$L(S) \equiv \Big\{ (x; y) \in \Re_+^{\mid S \mid} \times \Re_+^M \mid \qquad (3)$$

$$\exists \; (p; q) \in \Delta^{\mid S \mid M} \text{ s.t. } (x; y; p; q) \text{ is a Lindahl equilibrium for S } S \Big\}$$

The *separating price correspondence for* $\succeq_t$, $H^t : X \to \Delta^M$ is defined as:

$$H^t(x; y) \equiv \big\{ (p; q) \in \Delta^M \mid \qquad (4)$$

$$(p; q)(x; y) < (p; q)(x'; y') \; \forall \; (x'; y') \in X \text{ s.t. } (x'; y') \succ_t (x; y) \big\} \, .$$

Note that the range of H^t is Δ^M. This means that the prices given by H^t cannot be the same prices faced by the agent at a Lindahl equilibrium for a coalition S since these prices are elements of $\Delta^{\mid S \mid M}$. However, if $(x; y)$ is a Lindahl allocation, then there must exist normalization factors $k^{r,t} \in (0, 1]$ for each agent that can be used to put the prices back into the $\Delta^{\mid S \mid M}$. This normalization is described in detail below.

An allocation $(x; y)$ for a coalition S is said to be *S-optimal* if it is feasible for S, and there does not exist a feasible Pareto dominant allocation $(\tilde{x}; \tilde{y})$ for S. Formally, $(x; y)$ is S-optimal if

$$\left(\sum_{\{r,t\} \in S} \left[x^{r,t} - \omega^t \right]; y \right) \in Y,$$

$$\nexists\, (\tilde{x}; \tilde{y}) \in \Re_+^{\mid S \mid} \times \Re_+^M \text{ s.t. } \left(\sum_{\{r,t\} \in S} \left[\tilde{x}^{r,t} - \omega^t \right]; \tilde{y} \right) \in Y,$$

$$\forall\, \{r, t\} \in S,\ (\tilde{x}^{r,t}; \tilde{y}) \succeq_t (x^{r,t}; y),$$

and

$$\exists\, \{r, t\} \in S \text{ s.t. } (\tilde{x}^{r,t}; \tilde{y}) \succ_t (x^{r,t}; y).$$

It will frequently be convenient to take advantage of the fact that the well known *Samuelson conditions* must be satisfied by any S-optimal allocation. This may be stated in the notation of this paper in the following way. If an allocation $(x; y)$ for a coalition $S \in G^R$ is S-optimal, then

$$\forall\, \{r, t\}, \{r', t'\} \in S,\ \exists\, (p^{r,t}; q^{r,t}) \in H(x^{r,t}; y),\ \exists\, (p^{r',t'}; q^{r',t'}) \in H(x^{r',t'}; y),$$

$$\text{and } \exists\, k^{r,t}, k^{r',t'} \in (0, 1] \text{ s.t. } k^{r,t} \times p^{r,t} = k^{r',t'} \times p^{r',t'} \equiv p,$$

$$(p; \ldots, k^{r,t} \times q^{r,t}, \ldots, k^{r',t'} \times q^{r',t'} \ldots,) \in \Delta^{\mid S \mid M},$$

and

$$\exists\, (\tilde{p} : \tilde{q}) \in MC \left(\sum_{\{r,t\} \in G^R} [x^{r,t} - \omega^t]; y \right) \text{ s.t. } \forall\, m \in \mathcal{M},\ \sum_{\{r,t\} \in S} \frac{q_m^{r,t}}{p^{r,t}} = \frac{\tilde{q}_m}{\tilde{p}_m}.$$

Note that the normalization factor cancels out in the last line.

The *offer correspondence*, $OC^t : \mathbf{R}_+^M \to \Re_+$, is defined as follows:

$$OC^t(y) = \Big\{ x \in \Re_+ \mid\ \exists\, (p; q) \in \Delta^M \text{ s.t.} \tag{5}$$

$$(p;q)(x;y) \le p\omega^t \text{ and } (x;y) \succeq_t (x';y') \ \forall\ (x';y') \in X \text{ s.t. } (p;q)(x';y') \le p\omega^t\}$$

The graph of this correspondence is the offer curve for an agent of type t. The reader may verify that $x \in OC^t(y)$ if and only if for some $(p;q) \in H^t(x;y)$, $(p;q)(x;y) = p\omega^t$.[3]

An allocation $(x;y)$ is in the *core* for the grand coalition G^R if,

$$\left(\sum_{\{r,t\} \in G^R} \left[x^{r,t} - \omega^t \right] ; y \right) \in Y,$$

$$\nexists\ S \subseteq G^R, \text{ and } (\tilde{x};\tilde{y}) \in \Re_+^{|\,S\,|} \times \Re_+^M \text{ s.t. } \left(\sum_{\{r,t\} \in S} \left[\tilde{x}^{r,t} - \omega^t \right] ; \tilde{y} \right) \in Y,$$

and

$$\forall\ \{r,t\} \in S, (\tilde{x}^{r,t};\tilde{y}) \succeq_t (x^{r,t};y),$$

and in addition:

$$\exists\ \{r,t\} \in S \text{ s.t. } (\tilde{x}^{r,t};\tilde{y}) \succ_t (x^{r,t};y).$$

This is just the usual requirement that no coalition can block any core allocation. Notice that blocking coalitions can only consume the public goods produced using their own resources. They cannot free ride off the public goods produced by the rest of the agents. This follows Foley and others. The set of core allocations for the grand coalition G^R will be denoted by $C(G^R)$.

In order to be assured that Lindahl equilibria exist, and are in the core, we will take advantage of Foley's (1970) method of modeling public goods as jointly produced private goods. If it can be shown that competitive equilibria exist, and are in the core for the associated private goods economy, then it may be concluded that Lindahl equilibria exist and are in the core of the original public goods economy. But it is then easy to verify that A1-A3, B1-B3 and C1 imply all of the assumptions used by McKenzie (1981) to show existence. Milleron's (1972) theorem 4.1 may then be used to conclude that the Lindahl equilibrium is always in the core. The exercise of explicitly carrying

[3] Also see equation 8 below

out the translation of this model into a private goods economy is not performed here because it is long, and it would essentially be a duplication of Foley's contribution.

In lemma 1, a useful relationship between the core, agent's offer curves and the Lindahl equilibria is shown.

Lemma 1. *Let $\mathcal{E}^R$ satisfy A1-A3, B1-B3 and C1. If $(x;y) \in C(G^R)$, and for all $\{r,t\} \in \mathcal{R} \times \mathcal{T}$, $x^{r,t} \in OC^t(y)$, then $(x;y) \in L(G^R)$.*

Proof/

1. If $\forall \ \{r,t\} \in \mathcal{R} \times \mathcal{T}$, $x^{r,t} \in OC^t(y)$, then by the definitions of H^t and OC^t $\forall \ (r,t) \in \mathcal{R} \times \mathcal{T}$, $\exists \ (p^{r,t};q^{r,t}) \in H^t(x^{r,t};y)$ such that:

$$(p^{r,t};q^{r,t})(x^{r,t};y) \leq p\omega^t$$

   and

$$(x^{r,t};y) \succeq_t (x';y') \ \forall \ (x';y') \in X \text{ s.t. } (p^{r,t};q^{r,t})(x';y') \leq p^{r,t}\omega^t.$$

2. Also, since core allocations are feasible by definition,

$$\left(\sum_{\{r,t\} \in G^R} (x^{r,t} - \omega^t);y \right) \in Y.$$

3. Finally, suppose

$$\forall \ \{r,t\} \in G^R, \text{ and } \forall \ (p^{r,t};q^{r,t}) \in \Delta^M \text{ s.t. :}$$

   a. $(p^{r,t};q^{r,t}) \in H^t(x^{r,t};y)$,
   b. $(p^{r,t}q^{r,t})(x^{r,t};y) \leq p^{r,t}\omega^t$,
   c. $(x^{r,t};y) \succeq_t (x';y') \ \forall \ (\hat{x};\hat{y}) \in X$, s.t. $(p^{r,t};q^{r,t})(\hat{x};\hat{y}) \leq p^{r,t}\omega^t$, it is not the case that

$$\exists \ (p^{r,t};q^{r,t}) \in H(x^{r,t};y), \ \exists \ (p^{r',t'};q^{r',t'}) \in H(x^{r',t'};y), \ \exists \ k^{r,t} \in (0,1] \text{ and } \exists \ p \mid$$

$$k^{r,t} \times p^{r,t} = p \text{ and } (p;k^{1,1} \times q^{1,1},\ldots,k^{R,T} \times q^{R,T}) \in \Delta^{\,|\,S\,|\,M},$$

and,

$$\forall\ (\hat{z};\hat{y}) \in Y,$$

$$\left(p;\sum_{\{r,t\}\in G^R} q^{r,t}\right)\left(\sum_{\{r,t\}\in G^R}[x^{r,t}-\omega^t];y\right) \geq$$

$$\left(p;\sum_{\{r,t\}\in G^R} q^{r,t}\right)(\hat{z};\hat{y})\,.$$

But this would violate the Samuelson conditions. This in turn contradicts the hypothesis that $(x;y) \in C(G^R)$ and is therefore Pareto optimal.

■

The core is said to converge if the condition of lemma 1 is approximately satisfied in a way which becomes more exact as the economy grows without bound. A sequence of core allocations $\{(x^R;y^R)\}$, where for all R, $(x^R;y^R) \in C(G^R)$, is said to "ϵ-close" to the Lindahl allocations, if only a vanishing proportion of the agents in G^R remain off their offer curves by at least ϵ as R grows without bound. Define the coalition $S(\epsilon, x; y)$ to be the set of agents who are off there offer curves by ϵ at the allocation $(x;y)$.

$$S(\epsilon,x;y) \equiv \left\{\{r,t\} \in G^R \text{ if } \forall\ \tilde{x}^{r,t} \in OC^t(y),\ x^{r,t} \geq \tilde{x}^{r,t}+\epsilon,\ \text{or } x^{r,t} \leq \tilde{x}^{r,t}-\epsilon\right\}. \quad (6)$$

Then let the function $NOC : \Re_+ \times \Re_+^{RT} \times \Re_+^M \to [0,1]$, give the proportion of displaced agents in $\mathcal{E}^R$.

$$NOC(\epsilon,x;y) \equiv \frac{\mid S(\epsilon,x;y)\mid}{RT}. \quad (7)$$

The core is said to *converge* to the set of Lindahl allocations if for any sequence of core allocations $\{(x^R;y^R)\}$, and for any $\epsilon > 0$,

$$\lim_{R\to\infty} NOC(\epsilon,x^R;y^R) = 0.$$

In words, the core is said to converge to the set of Lindahl equilibria if for every allocation in the core of a sufficiently large economy, only a vanishingly small proportion of agents pay anything even slightly different from their Lindahl tax for whatever Pareto optimal quantity of public goods is produced by the grand coalition.

It is essential to the lemmas in sections three and four that the correspondence OC^t be well defined. The purpose of lemmas 2 through 6 is to show that under assumptions A1-A3 and C1, $OC^t(y) \neq \emptyset$ for any $y \in \mathbf{R}^M_+$

Lemma 2. *Let $\succeq_t$ satisfy A1-A3. Then for all $(x;y) \in X$, $H^t(x,y) \neq \emptyset$.*

Proof/

Consider the upper contour set of $(x;y) \in X$ for $\succ_t$:

$$U^t(x;y) \equiv \{(x';y') \in X \mid (x';y') \succeq_t (x;y)\}.$$

By A1-A3, $U^t(x;y) \neq \emptyset$ and is also closed and convex. Then by the Hahn-Banach theorem, for all $(x;y) \in X$,

$$\exists\ (p;q) \in H^t(x,y).$$

■

Lemma 3. *Let $\succeq_t$ satisfy A1-A3. Then H^t is an UHC correspondence.*

Proof/

Let $\{(x^\nu;y^\nu)\}$ be a convergent sequence in X, let $\lim_{\nu\to\infty}, (x^\nu;y^\nu) = (\tilde{x};\tilde{y})$, and let $(p^\nu;q^\nu) \in H^t(x^\nu;y^\nu)$ for all ν. Note that $(p^\nu;q^\nu)$ exists by lemma 2, and since $(p^\nu;q^\nu) \in \Delta^M$ for all ν, there exists a subsequence $\{(p^{\nu'};q^{\nu'})\}$ such that $\lim_{\nu'\to\infty}(p^{\nu'};q^{\nu'}) = (\tilde{p};\tilde{q}) \in \Delta^M$. It must be shown that

$$(\tilde{p};\tilde{q})(\tilde{x};\tilde{y}) < (\tilde{p};\tilde{q})(x';y')\ \forall\ (x';y') \in X \text{ s.t. } (x';y') \succ_t (\tilde{x};\tilde{y}).$$

Suppose not, then

$$\exists\ (x';y') \in X \text{ s.t. } (x';y') \succ_t (\tilde{x};\tilde{y})$$

and

$$(\tilde{p};\tilde{q})(\tilde{x};\tilde{y}) \geq (\tilde{p};\tilde{q})(x';y').$$

By continuity of preferences,

$$\exists\ \epsilon > 0 \text{ s.t. } (x' - \epsilon;y') \succ_t (\tilde{x};\tilde{y})$$

$$\text{and } (\tilde{p};\tilde{q})(\tilde{x};\tilde{y}) > (\tilde{p};\tilde{q})(x'-\epsilon;y').$$

But then by continuity of preferences, for sufficiently large ν',

$$(x'-\epsilon;y') \succ_t (x^{\nu'};y^{\nu'}).$$

By the definition of H^t, for sufficiently large ν',

$$(p^{\nu'};q^{\nu'})(x^{\nu'};y^{\nu'}) < (p^{\nu'};q^{\nu'})(x'-\epsilon;y').$$

But,

$$(p^{\nu'};q^{\nu'}) \to (\tilde{p};\tilde{q}), \text{ and } (x^{\nu'};y^{\nu'}) \to (\tilde{x};\tilde{y})$$

so

$$(\tilde{p};\tilde{q})(\tilde{x};\tilde{y}) \leq (\tilde{p};\tilde{q})(x'-\epsilon;y'),$$

a contradiction.

■

Lemma 4. *Let $\succeq_t$ satisfy A1-A3. Then H^t is convex-valued.*

Proof/

Suppose not. Then

$$\exists\, (x;y) \in X,\ \exists\, (p;q),(p';q') \in H^t(x;y) \text{ and } \exists\, \lambda \in [0,1] \text{ s.t.}$$

$$\lambda(p;q) + (1-\lambda)(p';q') \equiv (p'';q'') \notin H^t(x;y).$$

Then,

$$\exists\, (\tilde{x};\tilde{y}) \in X \text{ s.t. } (\tilde{x};\tilde{y}) \succ_t (x;y) \text{ and } (p'';q'')(x;y) \geq (p'';q'')(\tilde{x};\tilde{y}),$$

which implies,

$$\lambda\left\{(p;q)\left[(x;y)-(\tilde{x};\tilde{y})\right]\right\} + (1-\lambda)\left\{(p';q')\left[(x;y)-(\tilde{x};\tilde{y})\right]\right\} \geq 0.$$

This, however, is impossible since

$$(p;q) \in H^t(x;y) \text{ and } (p';q') \in H^t(x;y).$$

■

Now define the map $f_y^t : [0, \omega^t] \subset \Re_+ \to \Re$ by:

$$f_y^t(x) \equiv \{\gamma \in \Re \mid \exists\, (p;q) \in H^t(x;y) \text{ and } \gamma = (p;q)(x - \omega^t; y)\}. \tag{8}$$

This correspondence f_y^t gives the values of excess demand at each of the prices supporting $(x;y)$. More exactly, it is the differences between the possible values of the endowment that an agent of type t would have to have, and the value of the one he actually has, if $(x;y)$ were to be on his offer curve. Thus, for a particular value of y, if $0 \in f_y^t(x)$, then $(x;y)$ is on the offer curve of agent of type t.

Lemma 5. [4] *Under assumptions A1-A3 and C1, for all* $(\omega^t; y) \in X$, *there exists* $x \in [0, \omega^t]$ *such that* $0 \in f_y^t(x)$.

Proof/

Clearly f_y^t is UHC, convex-valued, and non-empty valued since it is the product of correspondences with these properties. Consider the following subsets of the domain:

$$D^+ \equiv \{x \in [0, \omega^t] \mid \exists\, z \in f_y^t(x) \text{ and } z \geq 0\}$$

$$D^- \equiv \{x \in [0, \omega^t] \mid \exists\, z \in f_y^t(x) \text{ and } z \leq 0\}$$

1. Note first that $D^- \neq \emptyset$. Let $(p;q) \in H^t(0;y)$. Then

$$\forall\, (x';y') \in X \text{ s.t. } (x';y') \succ_t (0;y), (p;q)(0;y) < (p;q)(x';y').$$

Then, since $qy < px' + qy'$,

$$\forall\, \lambda \in (0,1],\ \lambda qy < \lambda px' + \lambda qy'.$$

Then since

$$\lambda p < p + (1-\lambda)(1-p)$$

$$\lambda qy < (p + (1-\lambda)(1-p))x' + \lambda qy'.$$

But,

$$p + (1-\lambda)(1-p) + \lambda \sum_{i=1}^{M} q_i = p + (1-\lambda)(1-p) + \lambda(1-p) = 1.$$

Therefore,

$$(p + (1 - \lambda(1-p); \lambda q) \in H^t(0; y).$$

However,by strict monotonicity of preferences in the private good, $p > 0$, and so

$$\lim_{\lambda \to \infty} p + (1-\lambda)1 - p) = 1.$$

But by assumption C1, $\omega^t > 0$. It follows that for small enough λ,

$$(p + (1-\lambda)(1-p))(0 - \omega^t) + \lambda qy \leq 0.$$

Thus $0 \in D^-$, and so $D^- \neq \emptyset$.

2. Also note that $D^+ \neq \emptyset$. By monotonicity of preferences,

$$\forall\ (x; y) \in X \text{ and } \forall\ (p; q) \in H^t(x; y),\ (p; q) > 0.$$

So

$$p(\omega^t - \omega^t) + qy \geq 0.$$

Thus $\omega^t \in D^+$, and so $D^+ \neq \emptyset$.

3. Finally note that $0 \notin f_y^t(x)$ $x \in [0, \omega^t]$ implies $D^+ \cap D^- = \emptyset$. This follows from the fact that f_y^t is convex-valued which implies that for any $x \in [0, \omega^t]$, if $x \in D^+$ and $x \in D^-$, then $0 \in f_y^t(x)$.
4. It only remains to show that $D^+ \cap D^- \neq \emptyset$. Suppose not. Then since both sets are non-empty, at most one of them can be closed. Without loss of generality, suppose D^+ is not closed. Then there exists a sequence, $\{x^\nu\}$, in D^+ such that

$x^\nu \to x$, $0 \le f_y^t(x^\nu)$ and $0 > f_y^t(x)$. But this contradicts the UHC of f_y^t on $[0, \omega^t]$. Therefore, there is some $x \in \mathbf{R}_+^N$ such that $0 \in f_y^t(x)$.

■

Lemma 6. *For all* $y \in \mathbf{R}_+^M$, $OC^t(y) \neq \emptyset$.

Proof/

From lemma 5,

$$\forall\ (\omega^t; y) \in X,\ \exists\ x \in [0, \omega^t] \text{ s.t. } 0 \in f_y^t(x).$$

But then

$$\exists\ (p; q) \in H^t(x; y) \text{ s.t. } (p; q)(x - \omega^t; y) = 0.$$

Therefore, by definition of H^t :

$$\exists\ (p; q) \in H^t(x; y) \text{ s.t. } (p; q)(x; y) \le p\omega^t$$

and

$$\forall\ (x'; y') \in X \text{ s.t. } (x'; y') \succeq_t (x; y), (p; q)(x'; y') > p\omega^t.$$

Thus $x \in OC^t(y)$

■

3. Two Convergence Theorems

Two separate convergence theorems are proved in this section. The first shows that if for all agents, the value in terms of private good of receiving a given multiple of their current public goods consumption level goes to zero as the level of public goods goes to infinity, then the core converges to the set of Lindahl allocations. This assumption is called asymptotic satiation, and it is easy to see why it drives the result.

The reason that the core of a public goods economy is typically quite large is that the grand coalition is able to spread the costs of public goods production more widely than any subcoalition. As a consequence, subcoalitions find it difficult to block allocations offered by the grand coalition even when all the agents in the subcoalition are charged more than their Lindahl taxes. The larger bundle of public goods that the grand coalition is able to offer can compensate agents for substantial losses in private good consumption. But if all agents become asymptotically satiated in public good, then this ceases to be true. Now agents do not care very much about extremely high levels of public goods consumption. This puts a limit on the degree to which the grand coalition can "exploit" its cost advantage. Suppose, for example, that the grand coalition tried to give even a small proportion of the agents an level of private good consumption that was less than the level at a Lindahl allocation. As the economy got large, the members of the subcoalition would find that they are sufficiently numerous that they could easily produce a bundle of public goods that is almost as good as the one offered by the grand coalition. The subcoalition would then be able to block the effort to give less private good than they would get at a Lindahl allocation. In short, asymptotic satiation removes the increasing returns to coalitional size that are embedded in the public goods technology. As a result, even small coalitions are able to block attempts on the part of the grand coalition to deviate from the Lindahl allocations.

Before giving a formal definition of asymptotic satiation, consider the notion of *compensating variation.* This will be used to measure the difference in desirability between two given bundles of public and private goods. Specifically, the compensating variation is the largest amount of the private good that could be taken away from bundle 2 while still leaving the agent at least as well off at bundle 1. The compensating variation can be positive or negative. It is always well defined if it is non-negative (or equivalently, the second bundle is better than the first). Formally, for an agent of type t:

$$CV^t(x; y; x'; y') \equiv \{\max z \in \mathbf{R} \mid (x; y) \preceq_t (x' - z; y')\}. \tag{9}$$

Now asymptotic satiation can be defined as follows:

A4) $\forall\, x \in \Re_+,\ \forall\, \sigma \in (0,1],$ and $\forall\, t \in \mathcal{T}, \lim_{\parallel y \parallel \to \infty} CV^t(x;\sigma y;x;y) = 0.$

Examples of preferences that satisfy asymptotic satiation include those for which consumers are eventually satiated in public good. To give only two:

$$u(x;y) = \min[x\ ;\ y]$$

and

$$u(x;y) = \begin{cases} xy & \text{if } y \leq y^* \\ xy^* & \text{if } y \geq y^* \end{cases}.$$

An example of preferences which satisfy asymptotic satiation, but which do not exhibit satiation in public goods can be obtained by picking any level set of a Cobb-Douglas map, say $xy = 1$ and translating it up and down the 45 degree line to get a complete family of indifference curves:

$$u(x;y) = -x - y + \sqrt{(x-y)^2 + 4}.$$

In these examples, the offer curves are asymptotic to ω^t as y goes to infinity. In other words, the Lindahl taxes go to zero as the bundle of public goods gets large. Lemma 7 shows that this is a general implication of asymptotic satiation. This means, incidentally, that Cobb-Douglas preferences do not satisfy asymptotic satiation. It is not hard to confirm this by noticing that the compensating variation between two bundles as described in assumption 4 remains constant.

Lemma 7. *Let $\mathcal{E}^R$ satisfy assumptions A1-A4, B1-B3, and C1. Then for all $t \in \mathcal{T}$,*

$$\lim_{\parallel y \parallel \to \infty} OC^t(y) = \omega^t.$$

<u>Proof/</u>

Suppose not. Then

$$\exists\, \{y^\nu\} \text{ s.t. } \parallel y^\nu \parallel\ \to \infty \text{ and } \exists\, \epsilon > 0 \text{ s.t. } \forall\, \nu',\ \exists\, \nu \geq \nu' \text{ and } \exists\, x^\nu \in OC^t(y^\nu)$$

$$\text{s.t. } x^\nu \leq \omega^t - \epsilon.$$

Note that the possibility that $x^\nu \geq \omega^t + \epsilon$ need not be considered since by monotonicity all prices are non-negative and so $x^\nu \leq \omega^t$. Also by monotonicity, for all ν, and for all $x^\nu \in OC^t(y^\nu)$

$$\exists\ (p^\nu; q^\nu) \in H^t(x^\nu; y^\nu) \text{ s.t. } q^\nu y^\nu = p^\nu(\omega^t - x^\nu).$$

But since,

$$(p^\nu; q^\nu)(\tfrac{1}{2}\omega^t + \tfrac{1}{2}x^\nu; \tfrac{1}{2}y^\nu) = p^\nu \omega^t,$$

it follows that

$$(\tfrac{1}{2}\omega^t + \tfrac{1}{2}x^\nu; \tfrac{1}{2}y^\nu) \preceq_t (x^\nu; y^\nu).$$

Recall that asymptotic satiation implies that

$$\forall\ x \in \mathbf{R}_+, \text{ and } \forall\ t \in \mathcal{T}, \lim_{\|\ y^\nu\ \| \to \infty} CV^t(x; \tfrac{1}{2}y^\nu; x; y^\nu) = 0.$$

So by monotonicity, for large ν

$$(\tfrac{1}{2}\omega^t + \tfrac{1}{2}x^\nu; \tfrac{1}{2}y^\nu) \succ_t (\tfrac{1}{2}\omega^t + \tfrac{1}{2}x^\nu - \tfrac{1}{2}\epsilon; y^\nu)$$

But by hypothesis,

$$\omega^t - \epsilon \geq x^\nu$$

and therefore,

$$\tfrac{1}{2}\omega^t + \tfrac{1}{2}x^\nu - \tfrac{1}{2}\epsilon \geq x^\nu.$$

So by monotonicity,

$$(\tfrac{1}{2}\omega^t + \tfrac{1}{2}x^\nu - \tfrac{1}{2}\epsilon; y^\nu) \succeq_t (x^\nu; y^\nu).$$

But then by transitivity,

$$(\tfrac{1}{2}\omega^t + \tfrac{1}{2}x^\nu; \tfrac{1}{2}y^\nu) \succ_t (x^\nu; y^\nu),$$

which is a contradiction.

■

One additional technical assumption will be used to prove this convergence theorem.

A5) There exists $\theta > 0$ such that for all $t \in \mathcal{T}$, all $(x;y) \in X$, and all $\hat{x} \geq x$, if for all $(p;q) \in H^t(x;y)$ such that $\sum_{m \in \mathcal{M}} \frac{q_m}{p} \geq \bar{q}$, then for all $(\hat{p};\hat{q}) \in H^t(\hat{x};y)$, $\sum_{m \in \mathcal{M}} \frac{\hat{q}_m}{\hat{p}} \geq \theta \bar{q}$.

This assumption says that if the total of the marginal willingness to pay for all types of public good for an agent of type t ($TMWP^t$) is larger than some $\bar{q}$ at an allocation $(x;y)$, then if additional private good is given to the agent, there is a multiplicative limit θ on how much the $TMWP^t$ can go down. Assumption A5 can be interpreted as allowing, but limiting the degree to which the marginal rate of substitution of private good for public goods can increase as the consumption of public goods increases. This may seem like an odd requirement, but in reality it is only a weakening of better known assumptions. For example, if preferences are assumed to be quasi-linear in the private good, then $\theta = 1$. Alternatively if the ordinary assumption of diminishing marginal rates of substitution in the private good is made, then $\theta \geq 1$. Thus, Assumption A6 is not very restrictive.

Theorem 1. *Let $\{\mathcal{E}^R\}$ be any sequence of economies satisfying A1-A5, B1-B3 and C1. Then the core converges to the set of Lindahl allocations.*

Proof/

Suppose not. Then there exists a sequence of allocations, $\{(x^R;y^R)\}$, such that for all R, $(x^R;y^R) \in C(G^R)$ and

$$\exists\, \epsilon > 0 \text{ and } \exists\, \alpha \in (0,1] \text{ s.t. } \forall\, R' \,\exists\, R > R', NOC(\epsilon, x^R; y^R) \geq \alpha.$$

Let S be coalition of agents who are off their offer curves by at least ϵ. Divide S into two subcoalitions: S^{R^+}, consisting of the agents who are above their offer curves by ϵ or more (and are therefore paying less than their Lindahl taxes), and S^{R^-}, consisting of the agents who are below their offer curves by ϵ or more (and are therefore paying more than their Lindahl taxes). Formally,

$$S^{R^+} \equiv \{\{r,t\} \in G^R \mid \forall\, \tilde{x}^{r,t} \in OC^t(y^R), x^{r,t} \geq \tilde{x}^{r,t} + \epsilon\} \tag{10}$$

and

$$S^{R^-} \equiv \{\{r,t\} \in G^R \mid \forall\, \tilde{x}^{r,t} \in OC^t(y^R), x^{r,t} \leq \tilde{x}^{r,t} - \epsilon\}. \tag{11}$$

Also define the sequence of allocations $\{(x^{R^-}; y^{R^-})\}$ for the sequence of coalitions S^{R^-} where $x^{R^-,r,t} \equiv \omega^t - \frac{1}{2}\epsilon$ for $\{r,t\} \in S^{R^-}$ and y^{R^-} is the largest feasible vector of public goods when all the tax revenue collected by S^{R^-} is spent on a bundle proportional to y^R. Four cases are considered.

1. First, suppose

$$\parallel y^R \parallel\ \rightarrow \infty \text{ and } \forall\, R',\ \exists\, R > R' \text{ s.t. } \frac{\mid S^{R^-} \mid}{RT} \geq \tfrac{1}{2}\alpha.$$

Then there exists an arbitrarily large R such that S^{R^-} can block $(x^R; y^R)$ with the allocation $(x^{R^-}; y^{R^-})$. To see this note that the most that G^R can collect in taxes is $TR\omega^{\max}$ where $\omega^{\max}$ is the largest endowment held by any agent of any type. On the other hand, for all R', there exists $R > R'$ such that the tax revenue collected by S^{R^-} is at least collects $\frac{1}{2}\alpha TR\frac{1}{2}\epsilon$. But Y is a convex cone with zero origin, and the tax revenue of the coalition S^{R^-} is always at least a fraction, σ, of G^R's for arbitrarily large R where

$$\sigma \equiv \frac{\frac{1}{2}\alpha RT\frac{1}{2}\epsilon}{TR\omega^{\max}} = \frac{\frac{1}{4}\alpha\epsilon}{\omega^{\max}},$$

and so $y^{R^-} \geq \sigma y^R$. Thus,

$$\forall\, \{r,t\} \in S^{R^-}, (x^{R^-,r,t}; y^{R^-}) \succeq_t (\omega^t - \tfrac{1}{2}\epsilon; \sigma y^R). \tag{12}$$

But by asymptotic satiation:

$$\forall\, \{r,t\} \in G^R, \lim_{\parallel y^R \parallel \rightarrow \infty} CV^t(x^{R,r,t}; \sigma y^R; x^{R,r,t}; y^R) = 0.$$

By lemma 7,

$$\forall\, t \in \mathcal{T}, \lim_{\parallel y \parallel \rightarrow \infty} OC^t(y) = \omega^t.$$

Therefore, for large R,

$$\forall\, \{r,t\} \in S^{R^-}, \omega^t - \epsilon \geq x^{R,r,t}. \tag{13}$$

Also by asymptotic satiation for large R,

$$\forall \{r,t\} \in S^{R^-}, (\omega^t - \tfrac{1}{2}\epsilon; \sigma y^R) \succ_t (\omega^t - \epsilon; y^R). \tag{14}$$

So by (12), (13), and (14),

$$(x^{R^-,r,t}; y^{R^-}) \succeq_t (\omega^t - \tfrac{1}{2}\epsilon; \sigma y^R) \succ_t (\omega^t - \epsilon; y^R) \succeq_t (x^{R,r,t}; y^R). \tag{15}$$

Therefore, S^{R^-} blocks $(x^R; y^R)$ with $(x^{R^-}; y^{R^-})$ for large R

2. Next suppose

$$\exists\, y' \in \Re_+ \text{ s.t. } \forall\, R, \ \parallel y^R \parallel\ \leq y', \text{ and } \exists\, R > R' \text{ s.t. } \frac{|\, S^{R^-}\,|}{RT} \geq \tfrac{1}{2}\alpha.$$

Obviously, S^{R^-} can block $(x^R; y^R)$ with $\{(x^{R^-}; y^{R^-})\}$ for large R. This is because the S^{R^-} collects $\frac{1}{2}\alpha\frac{1}{2}\epsilon RT$ in taxes at $(x^{R^-}; y^{R^-})$ and clearly $\frac{1}{2}\alpha\frac{1}{2}\epsilon RT \to \infty$. Therefore, since by assumption, Y is a cone with zero origin, for large R, $y^R < y^{R^-}$. Also,

$$\forall \{r,t\} \in S^{R^-},\ x^{R^-,r,t} = \omega^t - \tfrac{1}{2}\epsilon > x^{R,r,t}.$$

In words, S^{R^-} can block $(x^R; y^R)$ with an allocation that gives each of its members more private good, and more public good. Thus no core allocation can place a fixed fraction of agents ϵ below their offer curves.

3. Now suppose

$$\parallel y^R \parallel\ \to \infty \text{ and } \forall\, R',\ \exists\, R > R' \text{ s.t. } \frac{|\, S^{R^+}\,|}{RT} \geq \tfrac{1}{2}\alpha.$$

But by lemma 7,

$$\forall\, t \in \mathcal{T} \lim_{\parallel y \parallel \to \infty} OC^t(y) = \omega^t.$$

Therefore, for large R,

$$\forall \{r,t\} \in S^{R^+}, x^{R,r,t} \geq \omega^t + \epsilon.$$

But then the agents in $G^R \setminus S^{R^+}$ can block $(x^R; y^R)$ by ejecting the agents in S^{R^+} from G^R and dividing among themselves the net payment of private goods that would have gone to these ejected agents. Therefore, no such allocation could be in the core.

4. Finally, suppose

$$\exists\, y' \in \Re^M_+ \text{ s.t. } \forall\, R,\ \| \, y^R \, \| \ \leq y'\ \forall\, R',\ \exists\, R > R' \text{ s.t. } \frac{|\, S^{R^+} \,|}{RT} \geq \tfrac{1}{2}\alpha.$$

First notice that

$$\forall\, \{r,t\} \in S^{R^+},\ \forall\, R, \text{ and } \forall\, \hat{x}^R \in OC^t(y^R),\ \hat{x}^R \leq \omega - \epsilon. \tag{16}$$

Otherwise, if an agent were above his offer curve by ϵ he would be receiving a net subsidy of private good. Then by the argument given in case 3, this could not be a core allocation.

But equation 16 implies

$$\forall\, \{r,t\} \in S^{R^+},\ \forall\, (\hat{p}^{r,t}; \hat{q}^{r,t}) \in H^t(\hat{x}^{R,r,t}; y^R), \sum_{m \in \mathcal{M}} \frac{\hat{q}^{r,t}_m}{\hat{p}^{r,t}} \geq \frac{\epsilon}{M \times y'} \equiv \bar{q}.$$

But

$$\forall\, \{r,t\} \in S^{R^+}, x^{R,r,t} \in [\hat{x}^{R,r,t}, \omega],$$

so by assumption 5,

$$\forall\, (p^{r,t}; q^{r,t}) \in H^t(x^{R,r,t}; y^R), \sum_{m \in \mathcal{M}} \frac{q^{r,t}_m}{p^{r,t}} \geq \theta \bar{q}.$$

But then for large R, and all $(p^{r,t}; q^{r,t}) \in H^t\left(x^{R,r,tz}\right)$,

$$\sum_{\{r,t\} \in S^{R+}} \sum_{m \in \mathcal{M}} \frac{q^{r,t}_m}{p^{r,t}} \geq \tfrac{1}{2}\alpha RT\theta\bar{q} > M\phi.$$

In words, the sum the agents' total marginal willingness to pay for all types of pubic goods exceeds the marginal cost of uniformly increasing the level of public goods. Therefore the Samuelson conditions do not hold at $(x^R; y^R)$, and so the allocation is not a core allocation as hypothesized.

■

The second convergence theorem says that if the marginal value of at least one of the public goods does not go to zero as the quantity of public goods goes to infinity, the core converges. This assumption is called *strict non-satiation* and it is even easier to see why it drives the result. For coalitions of sufficiently large size, strict non-satiation implies that every core allocation is one in which almost all of the private good is used to produce a Pareto optimal quantity of public goods. This obviously coincides with the Lindahl Equilibria.

As was noted in the introduction, this assumption is of less practical interest than asymptotic satiation. It implies that agents have such a strong preference for public activity that even for economies of moderate size, they choose to devote all of their resources to public goods. Perhaps this might reflect the tastes of members of utopian communities, but it these preferences are not shared by the majority of consumers. Formally, strict non-satiation is defined as:

A6) $\forall\ (x;y) \in X,\ \ \forall\ t \in \mathcal{T}$ and $\forall\ (p;q) \in H^t(x;y),\ \ \exists\ m \in \mathcal{M},$ and $\exists\ \beta > 0$ s.t. $q_m/p \geq \beta$.

An example of a class of preferences that satisfy strict non-satiation is any preference relation in which public goods and private goods are perfect substitutes.

Lemma 8. *Let $\{\mathcal{E}^R\}$ be any sequence of economies satisfying A1-A3, A6, B1-B3, and C1. Let $\{(x^R;y^R)\}$ be a sequence of allocations such that for all R, $(x^R;y^R) \in C(G^R)$. Finally, given $\epsilon > 0$, let $S_\epsilon^R \equiv \{\{r,t\} \in G^R \mid x^{R,r,t} \geq \epsilon\}$. Then,*

$$\forall\, \epsilon > 0,\ \lim_{R\to\infty} \frac{\mid S_\epsilon^R \mid}{RT} \equiv \alpha_\epsilon^R = 0.$$

Proof/

Suppose not. Then

$$\exists\ \{(x^R;y^R)\}\ \text{s.t.}\ \ \forall\ R, (x^R;y^R) \in C(G^R),\ \exists\ \epsilon > 0,\ \text{and}\ \exists\ \bar{\alpha} \in (0,1]$$

$$\text{s.t.}\ \ \forall\ R',\ \exists\ R \geq R'\ \text{s.t.}\ \alpha_\epsilon^R \geq \bar{\alpha}.$$

Let

$$S_\epsilon^{R,t} \equiv \{\{r,j\} \in S_\epsilon^R \mid j = t\}.$$

Then, since there are only a finite number of agents types,

$$\exists\, t' \in \mathcal{T} \text{ and } \exists\, \hat{\alpha} \in (0,1] \text{ s.t. } \forall\, R',\ \exists\, R \geq R',\ \text{s.t. } \frac{\mid S_{\epsilon}^{R,t'} \mid}{RT} \equiv \alpha_{\epsilon}^{R,t'} \geq \hat{\alpha}.$$

In words, at least $\hat{\alpha} \times RT$ agents of some type t' consume more than ϵ of the private good at some core allocation for an economy of arbitrarily large size.

Without loss of generality, suppose agents of type t' are strictly non-satiated in public good m. Then by strict non-satiation,

$$\forall\, (p^{r,t'}; q^{r,t'}) \in H^t(x^{R,r,t}; y^R),\ \frac{q_m^{r,t'}}{p_n^{r,t'}} \geq \beta > 0.$$

But since by monotonicity all prices are non-negative:

$$\forall\, R',\ \exists\, R \geq R' \text{ s.t. } \forall\, (p^{r,t}; q^{r,t}) \in H^t(x^{R,r,t}; y^R)\ \sum_{r \in \mathcal{R}} \sum_{t \in \mathcal{T}} \frac{q_m^{r,t}}{p^{r,t}} \geq \hat{\alpha} RT\beta.$$

However, by assumption B3,

$$\forall\, (z; y) \in Y, \text{ and } \forall\, (\tilde{p}; \tilde{q}) \in MC_m(z; y),\ \tilde{q}_m / \tilde{p} \leq \phi.$$

So clearly, for large enough R, $\phi < \hat{\alpha} RT\beta$. Thus, for large enough R, the Samuelson conditions are not satisfied at $(x^R; y^R)$. This contradicts the hypothesis that $\{(x^R; y^R)\}$ is a sequence of core allocations for G^R.

■

In words, it is impossible for a fixed fraction of agents to consume a positive quantity of the private good because, eventually, these agents become so numerous that even among themselves, the marginal benefit of spending their private good on public goods production exceeds the cost. Thus, no such allocation could be Pareto optimal, much less in the core.

Lemma 9. *Let $\{\mathcal{E}^R\}$ be any sequence of economies satisfying A1-A3, A6, B1-B3, and C1. Let $\{(x^R; y^R)\}$ be such that for all R, $(x^R; y^R) \in C(G^R)$. Then*

$$\lim_{R \to \infty} OC^t(y^R) = 0.$$

Proof/

Lemma 8 implies that

$$\lim_{R\to\infty} \sum_{\{r,t\}\in G^R} \omega^t - x^{r,t} = \infty.$$

Thus, for every $t \in \mathcal{T}$ there is a public good $m \in \mathcal{M}$ satisfying strict non-satiation for which

$$y_m^R \to \infty.$$

Then suppose,

$$\forall\, R',\ \exists\, R \geq R' \text{ s.t. } \exists\, t \in \mathcal{T}, \text{ and } \exists\, \epsilon \geq 0 \text{ and } \exists\, x \in OC^t(y^R) \text{ s.t. } x \geq \epsilon.$$

But then for large R, for agents of type t, since $(x^R; y^R) \in C(G^R)$, the Samuelson condition and Assumption B3 imply that

$$\forall\, r \in \mathcal{R},\ \exists\, (p^r; q^r) \in H^t(x^{R,r,t}; y^R), \text{ s.t. } \sum_{r\in\mathcal{R}} \frac{q_m^r}{p^r} \leq \phi.$$

But this is impossible by strict non-satiation since for large R,

$$\sum_{r\in R} \frac{q_m^r}{p^r} \geq R\beta > \phi.$$

But this contradicts the assumption of strict non-satiation.

■

Note that the two assumptions discussed in this section have exactly opposite implications for the behavior of the offer correspondence. Strict non-satiation implies that the offer correspondence converges to zero, while asymptotic satiation implies that it converges to the agent's endowment. The question core converges when the offer curve does neither of these things is taken up in the next section. First, it is shown that strict non-satiation is a sufficient condition for core convergence.

Theorem 2. *Under assumptions A1-A3, A6, B1-B3, and C1 the core converges to the set of Lindahl equilibria.*

Proof/

By lemma 9,

$$\forall \{(x^R;y^R)\} \text{ s.t. } \forall R(x^R;y^R) \in C(G^R), \lim_{R\to\infty} OC^t(y^R) = 0.$$

By lemma 8,

$$\forall \epsilon \geq 0 \lim_{R\to\infty} \alpha_\epsilon^R \equiv \frac{| S_\epsilon^R |}{RT} = 0.$$

Then since agents can't consume negative quantities of private goods it follows immediately that,

$$\epsilon > 0 \lim_{R\to\infty} NOC(\epsilon, x^R; y^R) = 0.$$

■

It is worth noting that core convergence also holds if some types of agents in the economy satisfy asymptotic satiation, and others satisfy strict non-satiation. Theorem 2 says that all almost agents who are strictly non-satiated are almost exactly on their offer curves for any core allocation of a large enough economy. Theorem 1 says the same for agents who are asymptotically satiated. Thus, if both types are in a single economy, then almost all of both types are almost exactly on their offer curves for every core allocation of a large enough economy. Therefore, the core converges.

4. Non-Convergence

This section explores the conditions under which the core can be shown not to converge to the set Lindahl equilibrium allocations. Attention is restricted to the class of one private good, one public good economies satisfying A1-A3, and B1-B3, and for which all consumers have smooth quasi-linear preferences. Formally, these two additional restrictions on the economy are:

A7) For all $t \in \mathcal{T}$, $\succ_t$ can be represented by a utility function of the form $u^t(x;y) = x + g^t(y)$
(Transferable utility)

A8) For all $t \in \mathcal{T}$, and all $(x;y) \in X$, if $(p;q) \in H^t(x;y)$ and $(p';q') \in H^t(x;y)$ then $(p;q) = (p';q')$.
(smoothness)

Note that assumptions A1-A3 imply that $g^t(y)$ is quasi-concave, continuous, and monotonic. Also note that assumption A7 implies that for any $(x;y) \in X$, if $(p;q) \in H^t(x;y)$ then $(p;q) \in H^t(x';y)$ for any $x' \geq 0$

Recall that strict non-satiation implies that agents' offer curves converge to zero as the level of public good goes to infinity. That is, agents pay almost all of their endowment of private good for high levels of public good consumption at every Lindahl allocation. Asymptotic satiation, on the other hand, implies that agents' offer curves converge to the agents' endowment. That is, agents pay almost nothing for high levels of public good consumption at every Lindahl allocation. The main theorem of this section shows that if neither one of these two conditions holds, then core is larger than the set of Lindahl allocations regardless of the size of the economy. More specifically, it will be shown that if for at least one agent type (without loss of generality, let this be agent type 1), the offer curve is strictly bounded from both zero and the endowment, then the core does not converge. This is in the spirit of negating the conditions that guaranteed convergence, but is logically unrelated. This condition is given formally below.

A9) There exists $\epsilon > 0$ and there exists$y' \geq 0$ such that if $y \geq y'$ then for all $x \in OC^1(y)$, $\omega_t - \epsilon \geq x \geq \epsilon$.

In addition we will make the technical assumption that agents of type 1 have strictly convex preferences.

A10) For all $(x;y), (x';y') \in X$ such that $y > y'$ and $(x;y) \sim_1 (x';y')$, if $(p;q) \in H^t(x;y)$

and $(p';q') \in H^t(x';y')$ then $q/p < q'/p'$.

(Strictly convex preferences for agents of type 1)

Lemma 10 uses these the assumptions of strict convexity and transferable utility to prove a useful fact about the set of supporting prices of type 1.

Lemma 10. *For all $\mathcal{E}^R$ that satisfy A1-A3, A7-A10, B1-B3 and C1, and for all $(x;y),(\bar{x};\bar{y}) \in X$ such that $y > \bar{y}$, if $(p;q) \in H^1(x;y)$ and $(\bar{p};\bar{q}) \in H^1(\bar{x};\bar{y})$ then $q/p < \bar{q}/\bar{p}$.*

Proof/

1. Suppose first that $(x;y) \sim_1 (\bar{x};\bar{y})$. Then assumption A10 can be used directly to conclude that the lemma is true.
2. Suppose next that $(x;y) \succ_1 (\bar{x};\bar{y})$. Then by transferable utility,

$$\exists\, x' > 0 \text{ s.t. } (x';\bar{y}) \sim_1 (x;y).$$

   But then the assumption of transferable utility implies

$$\text{if } (\bar{p};\bar{q}) \in H^1(\bar{x};\bar{y}) \text{ then } (\bar{p};\bar{q}) \in H^1(x';\bar{y}).$$

   Also by strict convexity, for agents of type 1,

$$\text{if } y > \bar{y}, (x';\bar{y}) \sim_1 (x;y), (\bar{p};\bar{q}) \in H^t(x';\bar{y}) \text{ and } (p;q) \in H^t(x;y),$$

   then,

$$\frac{q}{p} < \frac{\bar{q}}{\bar{p}}.$$

3. Finally, suppose that $(x;y) \prec_1 (\bar{x};\bar{y})$. Again, transferable utility implies that

$$\exists\, x' > 0 \text{ s.t. } (x';y) \sim_1 (\bar{x};\bar{y}).$$

   But then by transferable utility,

$$\text{if } (p;q) \in H^1(x;y) \text{ then } (p;q) \in H^1(x';y).$$

But then by strict convexity, for agents of type 1,

$$\text{if } y > \bar{y}, (x'; y) \sim_1 (\bar{x}; \bar{y}), (p; q) \in H^t(x; y), \text{ and } (\bar{p}; \bar{q}) \in H^t(\bar{x}; \bar{y})$$

then,

$$\frac{q}{p} < \frac{\bar{q}}{\bar{p}}.$$

■

A similar lemma can be proven about all agent types using transferable utility, smoothness, and convexity.

Lemma 11. *For $\mathcal{E}^R$ satisfying A1-A3, A7-A10, B1-B3 and C1, for all $t \in \mathcal{T}$, and all $(x; y)$, $(\bar{x}; \bar{y}) \in X$ such that $y \geq \bar{y}$, if $(p; q) \in H^t(x; y)$ and $(\bar{p}; \bar{q}) \in H^t(\bar{x}; \bar{y})$ then $q/p \leq \bar{q}/\bar{p}$.*

Proof/

By transferable utility and smoothness,

$$\forall\, t \in \mathcal{T},\ \forall\, x, \bar{x} \geq 0,\ \forall\, y \geq 0,$$

$$\text{if } (p; q) \in H^t(x; y) \text{ and }, (\bar{p}; \bar{q}) \in H^t(\bar{x}; y),\ \text{then } (p; q) = (\bar{p}; \bar{q}).$$

But by convexity and smoothness,

$$\forall\, t \in \mathcal{T},\ \text{if } (x; y) \sim_t (\bar{x}; \bar{y}), y \geq \bar{y}(p; q) = H^t(x; y) \text{ and }, (\bar{p}; \bar{q}) = H^t(\bar{x}; y),$$

then

$$\frac{q}{p} \leq \frac{\bar{q}}{\bar{p}}.$$

Therefore,

$$\forall\, t \in \mathcal{T} \text{ and } \forall\, x, \bar{x} \geq 0,\ \text{if } y \geq \bar{y},$$

$$\text{then } \forall\, (p; q) \in H^t(x; y) \text{ and } \forall\, (\bar{p}; \bar{q}) \in H^t(\bar{x}, \bar{y}),$$

$$\frac{q}{p} \leq \frac{\bar{q}}{\bar{p}}.$$

■

This entire section considers a one private, one public good economy satisfying A1-A3, A7-A10, B1-B3, and C1. Reference to this fact will be omitted in the statement of the propositions that follow. Let $\{(\bar{x}^R; y^R)\}$ denote a sequence of equal treatment Lindahl allocations. Formally, for all R

$$(\bar{x}^R; y^R) \in \{(x; y) \in L(G^R) \mid \forall\, t \in \mathcal{T} \text{ and } \forall\, r, r' \in \mathcal{R}, x^{r,t} = x^{r',t}\}. \tag{17}$$

Lemma 12 shows that such allocations exist.

Lemma 12. *Let $(x; y) \in L(G^R)$. Then $(\bar{x}; y) \in L(G^R)$ where for all $\{r, r\} \in \mathcal{R} \times \mathcal{T}$,*

$$\bar{x}^{r,t} = \frac{\sum_{r \in \mathcal{R}} x^{r,t}}{R}.$$

Proof/

First note that the allocation $(\bar{x}; y)$ is feasible since

$$R \sum_{t \in \mathcal{T}} [\omega^t - \bar{x}^t] = \sum_{\{r,t\} \in \mathcal{R} \times \mathcal{T}} [\omega^t - x^{r,t}],$$

and

$$\left(\sum_{\{r,t\} \in \mathcal{R} \times \mathcal{T}} [\omega^t - x^{r,t}]; y \right) \in Y.$$

Thus, to show that $(\bar{x}; y) \in L(G^R)$ it need only be shown $\bar{x}^t \in OC^t(y)$ for all $t \in \mathcal{T}$. So for particular type t, let

$$x^t_{\min} \equiv \min_{r \in \mathcal{R}} x^{r,t},$$

and

$$x^t_{\max} \equiv \max_{r \in \mathcal{R}} x^{r,t}.$$

Then by definition,

$$\exists\, (p; q_{\min}) \in H^t(x^t_{\min}; y) \text{ s.t. } (p; q_{min})(x^t_{\min} - \omega^t; y) = 0,$$

and

$$\exists\ (p; q_{\max}) \in H^t(x^t_{\max}; y) \text{ s.t. } (p; q_{max})(x^t_{\max} - \omega^t; y) = 0.$$

But then the assumption of transferable utility,

$$\forall\ x' > 0, (p; q_{\min}) \in H^t(x'; y) \text{ and } (p; q_{\max}) \in H^t(x'; y).$$

Also, by lemma 4, H^t is convex valued, and so

$$\forall\ \lambda \in [0,1], \text{ and } \forall\ x' \geq 0,\ \lambda(p; q_{\min}) + (1-\lambda)(p; q_{\max}) \in H(x'; y).$$

But then clearly,

$$\exists\ (p; \bar{q}) \in H^t(\bar{x}^t; y) \text{ s.t. } (p; \bar{q})(\bar{x}^t - \omega^t; y) = 0.$$

Thus, $\bar{x}^t \in OC^t(y)$.

■

Also note the following fact about any sequence of Lindahl allocations under these assumptions.

Lemma 13. *For all R, let $(x^R; y^R) \in L(G^R)$. Then*

$$\lim_{R\to\infty} y^R = \infty.$$

Proof/

1. First notice

$$\exists\ \epsilon > 0 \text{ s.t. } \forall\ y > 0,\ OC^1(y) \geq \epsilon.$$

This is immediate from assumption A9 for $y \geq y'$. So suppose

$$\exists\ y \in (0, y') \text{ s.t. } OC^1(y) = 0.$$

But then since $y > 0$,

$$\exists\ (p; q) \in H^1(\omega^1; y) \text{ s.t. } q = 0.$$

Then by lemma 10,

$$\forall\, x \geq 0 \text{ and } y'' > y,\ \exists\, (p;q) \in H^1(x;y''),\ \text{s.t. } q/p < 0.$$

But this is impossible since monotonicity implies that prices are positive.

2. Therefore, if $y > 0$, then for all R, the total Lindahl taxes collected by G^R at every Lindahl allocation is bounded from below by $RT\epsilon$. Then since the amount of taxes spent on public goods goes to infinity as the economy increases in size, and by assumption each unit of public good costs one unit of private good, $y^R \to \infty$.
3. It only remains to show that it is impossible that no public goods be produced at any Lindahl equilibrium for a sufficiently large economy. But lemma 10 and monotonicity imply that,

$$\forall\, x \geq 0, \text{ and } \forall\, (p;q) \in H^1(x;0),\ \exists\, \hat{q} > 0 \text{ s.t. } q/p > \hat{q}.$$

Otherwise for $y > 0$, $q < 0$ in violation of the assumption of monotonicity. But then for large R,

$$\forall\, t \in \mathcal{T},\ \forall\, x \geq 0,\ \forall\, (p;q) \in H^t(x;0), \text{ and } \forall\, (\tilde{p};\tilde{q}) \in MC(0;0)$$

$$\sum_{\{r,t\} \in G^R} \frac{q^{r,t}}{p} \geq R\hat{q} > \phi \geq \frac{\tilde{q}}{\tilde{p}}.$$

Thus, the Samuelson conditions are not satisfied for large R and so producing no public good could not be Pareto optimal, and therefore not a Lindahl allocation.

■

Now let us turn our attention to attempting to construct an allocation that is different from $(\bar{x}^R; y^R)$ and yet is still an element of the core. For even numbered replications, split the agents of type 1 evenly into two new types called 1^- and 1^+. Agents $\{r,1\}$ for $r = 1,\ldots,\frac{R}{2}$, will now be referred to as $\{r,1^-\}$ for $r = 1,\ldots,\frac{R}{2}$, and the agents $\{r,1\}$ for $r = \frac{R}{2}+1,\ldots,R$, will now be referred to as $\{r,1^+\}$ for $r = 1,\ldots,\frac{R}{2}$. Consider $(\bar{x}^R_\epsilon; y^R)$ as a possible element of the core where $\bar{x}^{R,r,1^-}_\epsilon =$

$\bar{x}^{R,1} - \epsilon$, for $r = 1, \ldots, \frac{R}{2}$, $\bar{x}_\epsilon^{R,r,1^+} = \bar{x}^{R,1} + \epsilon$, for $r = 1, \ldots, \frac{R}{2}$, and $\bar{x}_\epsilon^{R,r,t} = \bar{x}^{R,t}$ for $r \in \mathcal{R}$ and $t = 2, \ldots, T$. In words, $(\bar{x}_\epsilon^R; y^R)$ is an allocation that differs from the equal treatment Lindahl allocation $(\bar{x}^R; y^R)$ only in that all agents of type 1^- are taxed an extra ϵ units of private good, and agents of type 1^+ are taxed ϵ less. Note that by assumption A9, the offer curve of agents of type 1 are strictly bounded from both zero and the endowment by ϵ. Thus, all agents receive an amount of private good that is positive but less than ω^t at $(\bar{x}_\epsilon^R; y^R)$.

Lemma 14. *If a coalition S can block $(\bar{x}_\epsilon^R; y^R)$, then it must include some agents of type 1.*

Proof/

Suppose not. Then since agents of types $t = 2, \ldots, T$ are exactly as well off at $(\bar{x}_\epsilon^R; y^R)$ as they are at $(\bar{x}^R; y^R)$, if S can block $(\bar{x}_\epsilon^R; y^R)$ then S can also block $(\bar{x}^R; y^R)$. But $(\bar{x}^R; y^R)$ is a Lindahl equilibrium and therefore in the core. Thus no such S could block $(\bar{x}_\epsilon^R; y^R)$.

■

Lemma 15. *If a coalition S can block $(\bar{x}_\epsilon^R; y^R)$, then it must exclude some agents of type* 1.

Proof/

Suppose S is coalition that can block $(\bar{x}_\epsilon^R; y^R)$ with an allocation $(x^S; y^S)$ and includes all the agents of type 1. Note that agents of types $t = 2, \ldots, T$ are exactly as well off at $(\bar{x}_\epsilon^R; y^R)$ as they are in the $(\bar{x}^R; y^R)$, and so must be at least as well off at $(x^S; y^S)$ as at $(\bar{x}^R; y^R)$. Consider two cases:

1. Suppose

$$\exists \{r, 1^+\} \in S \text{ s.t. } x^{S,r,1^+} \leq \epsilon.$$

   Then since $(x^{S,r,1^+}; y^S) \succeq_1 (\bar{x}^{R,1} + \epsilon; y^R)$ by transferable utility, $(0; y^S) \succeq_1 (\bar{x}^{R,1}; y^R)$. But no agent can receive a negative allocation of private good. So the worst allocation that any agent of type 1 can receive in the coalition S is $(0; y^S)$. Thus all agents of type 1 are already at least as well off at $(x^S; y^S)$ as at $(\bar{x}^R; y^R)$. But then $(x^S; y^S)$ blocks $(\bar{x}^R; y^R)$, which is impossible since $(\bar{x}^R; y^R)$ is in the core.

2. Now suppose

$$\forall\, r = 1, \ldots, \frac{R}{2}, x^{S,r,1^+} \geq \epsilon$$

consider the allocation that obtains if for all $r = 1, \ldots, \frac{R}{2}$, agent $\{r, 1^+\}$ transfers ϵ private good to agent $\{r, 1^-\}$ By transferable utility, both types are at least as well off as at $(\bar{x}^R; y^R)$, and other agents are unaffected. The coalition S can therefore block $(\bar{x}^R; y^R)$ using this allocation. But again contradicts the fact that $(\bar{x}^R; y^R)$ is in the core.

■

Lemma 16. *If there exists a coalition $S \subseteq G^R$ that can block $(\bar{x}^R_\epsilon; y^R)$, then $(\bar{x}^R_\epsilon; y^R)$ can also be blocked by a coalition S' that contains all the agents of type 1^- and none of the agents of type 1^+*

Proof/

By lemma 14, S must contain some agents of type 1. Suppose first that

$$\forall\, \{r, 1\} \in S, x^{S,r,1} \geq \omega^t.$$

That is, suppose all agents of type 1 pay nothing or receive a net *subsidy* for being in S rather than contributing taxes to help produce public goods. Then clearly, it is possible to throw all agents of type 1 out of the coalition S, and still produce an allocation that blocks $(\bar{x}^R_\epsilon; y^R)$ by simply producing the same bundle of public goods, and distributing the subsidy that would have gone to the type 1's among the remaining agents. But this contradicts lemma 14 and so can not be the case.

Thus,

$$\exists\, \{r, 1\} \in S \text{ s.t. } x^{S,r,1} < \omega^t.$$

Suppose that this agent happens to be of type 1^+. Recalling that all 1^+'s are treated equally in the allocation $(\bar{x}^R_\epsilon; y^R)$, by monotonicity,

$$(\omega^1; y^S) \succeq_1 (x^{S,r,1}; y^S) \succeq_1 (\bar{x}^{R,1^+}_\epsilon; y^R) \succeq_1 (\bar{x}^{R,1^-}_\epsilon; y^R).$$

Thus, all agents of type 1 who are not already members S could be included in a larger coalition S' and be allowed to enjoy y^S public good while consuming their full endowment of private good. In other words, it is possible to admit all agents of type 1 not already in S to the coalition for free. These agents are at least as well off as they are at the allocation $(\bar{x}^R_\epsilon; y^R)$, and so this allocation over the coalition S' also blocks $(\bar{x}^R_\epsilon; y^R)$. But this contradicts lemma 15.

Therefore, no agent of type 1^+ can pay positive taxes. As a consequence it is possible to exclude them from the coalition S and still block. But then

$$\exists \{r, 1^-\} \in S \text{ s.t. } x^{S,r,1^-} < \omega^t.$$

Then following the argument above, recall that all 1^-'s are treated equally in the allocation $(\bar{x}^R_\epsilon; y^R)$. It follows by monotonicity that,

$$(\omega^1; y^S) \succeq_t (x^{S,r,1}; y^S) \succeq_t (\bar{x}^{R,1^-}_\epsilon; y^R).$$

Thus, all agents of type 1^- who are not already members S could be included in a larger coalition S' allowed to enjoy y^S public goods while consuming their full endowment of private good. These agent are at least as well off as they are at the allocation $(\bar{x}^R_\epsilon; y^R)$, and so this allocation over the coalition S' that includes all 1^- and excludes all 1^+ also blocks $(\bar{x}^R_\epsilon; y^R)$.

■

Lemma 17. *If there exist a coalition $S \subseteq G^R$ that can block $(\bar{x}^R_\epsilon; y^R)$, then $(\bar{x}^R_\epsilon; y^R)$ can also be blocked by a coalition S' that contains all the agents of type 1^-, no agents of type 1^+, and is S'-optimal.*

Proof/

By lemma 16, if S can block, then an S' can block that contains all the type 1^-'s and none of the type 1^+'s. Suppose that the blocking allocation over S' is not S'-optimal. Then take any Pareto superior and S'-optimal allocation over S'. Clearly, such an allocation must also block the grand coalition's allocation.

■

So far it has been shown that if $(\bar{x}^R_\epsilon; y^R)$ can be blocked, it can be blocked by a coalition of the type described in lemma 17. The next step is to exploit the fact that there exists a blocking coalition that contains no agents of type 1^+ to gain more information about this coalition and the associated blocking allocation.

Consider the question of what it would take to persuade all the agents of type 1^+ to join the blocking coalition. It would be certainly be possible to do so if each of these agents of type 1^+ could be made at least 2ϵ better off in terms of private good than agents of type 1^-. This is because agents of type 1^+ get exactly 2ϵ more private good than agents of type 1^- at $(\bar{x}^R_\epsilon; y^R)$. However, if S is to remain a blocking coalition, all the original agents must be at least as well off as they are at $(\bar{x}^R_\epsilon; y^R)$ after private good has been collected to "bribe" the agents of type 1^+ join S.

Thus the first question to answer for any such blocking coalition S is what is the maximum bribe in transferable utility terms that can be offered to each agent of type 1^+ in order to induce them to join the blocking coalition, and yet still leave all the original agents in S at least as well off as they are at the allocation $(\bar{x}^R_\epsilon; y^R)$. The next question, of course, is whether or not this bribe is sufficient. If the answer is yes, then there exists a coalition that includes all the type 1's, and that blocks $(\bar{x}^R_\epsilon; y^R)$. Since this is impossible by lemma 15, the original blocking coalition S could not have existed. Thus, $(\bar{x}^R_\epsilon; y^R)$ is in the core.

Consider the first question. There are two possible sources for this bribe. The first is any *surplus* transferable utility that exists at the blocking allocation. In other words, suppose that as much private good as possible is taken away from each agent in S while being careful to leave them at least as well off as they are at the allocation $(\bar{x}^R_\epsilon; y^R)$. This surplus could be divided evenly among the types 1^+ as a bribe to join S. Notice two things about this surplus. First, if a coalition blocks, the surplus associated with the blocking allocation must be positive. Second, since all agents of the same type (understanding that 1^+ and 1^- are different types) are treated equally in the allocation $(\bar{x}^R_\epsilon; y^R)$, they all must end up with identical allocations when all possible surplus is taken away from them. Formally, the surplus is defined as $sur : \Re^{\mid S \mid} \times \Re \times \Re^{RT} \times \Re \to \Re :$

$$sur((x; y), (\tilde{x}; \tilde{y})) \equiv \left(\frac{2}{R}\right) \sum_{\{r,t\} \in S} CV_t((\tilde{x}^{r,t}; \tilde{y}), (x^{r,t}; y)) \tag{18}$$

The second possible source for a bribe comes from the fact that types 1^+ may be allowed to join for free. That is, they may be allowed to join while paying no taxes, and consuming their endowment. Thus the difference between the consumption of private good by agents of 1^- after all potential surplus has been extracted from them, and the endowment of type 1's, measures how much better it is possible to make each agent of type 1^+ compared to agents of type 1^- by allowing the 1^+'s into S without paying any taxes. Call this the *bonus* utility. Formally, the bonus is defined as $bon : \Re^{\mid S \mid} \times \Re \times \Re^{RT} \times \Re \to \Re :$

$$bon((x; y), (\tilde{x}; \tilde{y})) \equiv \omega^1 - \{\min z \mid (z, y) \succeq_1 (\tilde{x}^{1^-}, \tilde{y})\} \tag{19}$$

Thus $bon((x; y), (\tilde{x}; \tilde{y})) + sur((x; y), (\tilde{x}; \tilde{y}))$ is the total amount of private good that can be given to each 1^+ in excess of what agents of type 1^- get in order to induce them to join the coalition S, while still leaving all the original agents in S at least as well off as at the allocation $(\bar{x}^R_\epsilon; y^R)$. The next few lemmas describe properties of these two correspondences on the assumption that $(\bar{x}^R_\epsilon; y^R)$ can be blocked with an S-optimal allocation by a coalition S that includes all the type 1^-'s and none of the type 1^+'s. Since by lemma 17, if $(\bar{x}^R_\epsilon; y^R)$ can be blocked by any coalition it can also be blocked by one having these properties, it is sufficient to show that the properties shown for this coalition lead to a contradiction.

Lemma 18. *For all R, and for any $S \in G^R$ that contains all types 1^- and no types 1^+ that can block $(\bar{x}^R_\epsilon; y^R)$, with an S-optimal allocation $(x^S; y^S)$,*

$$bon((x^S; y^S), (\bar{x}^R_\epsilon; y^R)) + sur((x^S; y^S), (\bar{x}^R_\epsilon; y^R)) \leq 2\epsilon.$$

Proof/

Suppose

$$bon((x^S;y^S),(\bar{x}^R_\epsilon;y^R)) + sur((x^S;y^S),(\bar{x}^R_\epsilon;y^R)) > 2\epsilon,$$

then there is enough excess private good to leave all the agents in S as well off as they are at the allocation $(\bar{x}^R_\epsilon;y^R)$ and yet offer each type 1^+ a level of consumption that leaves them 2ϵ better off than each type 1^-. But then types 1^+ are better off than at $(\bar{x}^R_\epsilon;y^R)$. So there exists a coalitions S that has all the type 1's, and blocks $(\bar{x}^R_\epsilon;y^R)$. But this contradicts lemma 15.

■

Lemma 19. *For all R, and for all $S \subset G^R$ that contains all types 1^- and no types 1^+ that can block $(\bar{x}^R_\epsilon;y^R)$, with an S-optimal allocation $(x^S;y^S)$,*

$$bon((x^S;y^S),(\bar{x}^R_\epsilon;y^R)) \geq sur((x^S;y^S),(\bar{x}^R_\epsilon;y^R)).$$

Proof/

Suppose instead that

$$bon((x^S;y^S),(\bar{x}^R_\epsilon;y^R)) = \epsilon' < \epsilon'' = sur((x^S;y^S),(\bar{x}^R_\epsilon;y^R)).$$

First notice that $\epsilon' > 0$. This is because if $\epsilon' \leq 0$ then at the allocation $(x^S;y^S)$, type 1^-'s pay no taxes, or may even be getting a net subsidy to be members of the blocking coalition S. But then, it is possible to form another blocking coalition without any type 1's at all by producing the same level of public good and distributing the net subsidy of private good that would have gone to the 1^-'s over the rest of the agents in S. This contradicts lemma 14. Also note that if S is able to block with $(x^S;y^S)$, then $sur((x^S;y^S),(\bar{x}^R_\epsilon;y^R)) > 0$. Thus by lemma 18, $\epsilon' + \epsilon'' \leq 2\epsilon$, so clearly $0 < \epsilon' \leq \epsilon$.

Now consider the allocation $(\bar{x}^R_{\epsilon-\epsilon'};y^R)$. This differs from $(\bar{x}^R_\epsilon;y^R)$ only in that $\epsilon - \epsilon'$ is taken from types 1^- and given to types 1^+ instead of ϵ. Agents of types $t = 2,\ldots,T$ are indifferent between the two allocations. But since

$$\bar{x}^{R,1^-}_{\epsilon-\epsilon'} = \bar{x}^{R,1^-}_\epsilon + \epsilon',$$

each agent of type 1^- will need to be compensated an additional $\epsilon' < \epsilon''$ in order to induce him to stay in the coalition S. Since by hypothesis, $\epsilon'' > \epsilon'$,

$$sur((x^S; y^S), (\bar{x}^R_{\epsilon-\epsilon'}; y^R)) = \epsilon'' - \epsilon' > 0.$$

In addition, the maximum contribution of the type 1^-'s to the production of y^S, the public goods level of the blocking coalition must go down by exactly ϵ'. Thus

$$bon((x^S; y^S), (\bar{x}^R_{\epsilon-\epsilon'}; y^R)) = 0.$$

But then it would be possible to get rid of all the type 1^-'s, since they contribute nothing to public goods production of the coalition S, and still have a blocking allocation over the remaining agents. But again, this is a contradiction of lemma 14.

■

Lemma 20. *For all $\epsilon' \in [0, \epsilon]$, if an S-optimal allocation $(x^S; y^S)$ for a coalition S that contains all types 1^- and no types 1^+ can block $(\bar{x}^R_{\epsilon'}; y^R)$ then it can also block $(\bar{x}^R_{\epsilon}; y^R)$.*

Proof/

Agents of type 1^- are better off at $(\bar{x}^R_{\epsilon'}; y^R)$ than at $(\bar{x}^R_{\epsilon}; y^R)$ since $\epsilon' \leq \epsilon$. All other agents in S are exactly as well off at these two allocations. Thus, if $(x^S; y^S)$ blocks $(\bar{x}^R_{\epsilon'}; y^R)$ then trivially, all agents are at least as well off at $(x^S; y^S)$ as at $(\bar{x}^R_{\epsilon}; y^R)$.

■

Lemma 20 is useful since it implies that only coalitions that can block $(\bar{x}^R_{\epsilon}; y^R)$ have the potential to block $(\bar{x}^R_{\epsilon'}; y^R)$ for any $\epsilon' \leq \epsilon$. As a consequence, all properties that have been shown about coalitions that block $(\bar{x}^R_{\epsilon}; y^R)$ must also hold for coalitions that block $(\bar{x}^R_{\epsilon'}; y^R)$. In lemma 21 the asymptotic behavior of such blocking coalitions is considered.

Lemma 21. *Suppose that for all $\epsilon' \in (0, \epsilon]$, there exists a sequence of coalitions, $\{S^R\}$, containing all agents of type 1^- and no agents of type 1^+, such that there exists R', such* *that for all $R \geq R'$ S^R can block the allocation $(\bar{x}^R_{\epsilon'}; y^R)$ with an S-optimal allocation $(x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'})$, then there also exists $\{(x^{S^R}; y^{S^R})\}$, a sequence of allocations for $\{S^R\}$, such that there exists R' such that for $R \geq R'$, S^R can block $(\bar{x}^R_{\epsilon}; y^R)$ with $(x^{S^R}; y^{S^R})$, and for which*

$$\lim_{R\to\infty} x^{S^R,1^-} = \omega^1.$$

Proof/

Suppose that for any $\epsilon' \in (0, \epsilon]$, such a sequence of blocking coalitions, $(x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'})$, exists. Recall that by lemma 18 for all R,

$$bon((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)) + sur((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)) \leq 2\epsilon'.$$

But by lemma 19,

$$bon((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)) \geq sur((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)),$$

and since the coalition blocks,

$$sur((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)) > 0.$$

Thus

$$2\epsilon' \geq bon((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)) + sur((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)) \geq 0.$$

also recall that by hypothesis, for any $\epsilon' \in (0, \epsilon]$ there exists a sequence of coalitions $\{S^R\}$ that contains all agents of type 1^- and no agents of type 1^+ and a sequence of S-optimal allocations $\{(x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'})\}$ that allow S^R to block $(\bar{x}^R_{\epsilon'}; y^R)$, for large enough R. But ϵ' can be chosen arbitrarily small. And since

$$2\epsilon' \geq bon((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)) + sur((x^{S^R}_{\epsilon'}; y^{S^R}_{\epsilon'}), (\bar{x}^R_{\epsilon'}; y^R)) \geq 0.$$

there must exist a sequence of allocations $\{(x^{S^R}; y^{S^R})\}$ that allow S^R to block $(\bar{x}^R_{\epsilon'}; y^R)$ for large R and for which

$$bon((x^{S^R}; y^{S^R}), (\bar{x}^R_{\epsilon'}; y^R)) \to 0.$$

But if $(x^{S^R}; y^{S^R})$ blocks $(\bar{x}^R_{\epsilon'}; y^R)$, then by lemma 20, $(x^{S^R}; y^{S^R})$ must also block $(\bar{x}^R_{\epsilon}; y^R)$. Then since

$$bon((x^{S^R}; y^{S^R}), (\bar{x}^R_{\epsilon'}; y^R)) \to 0,$$

it must also be true that

$$\lim_{R\to\infty} x^{S^R,1^-} \to \omega^1,$$

and $\{(x^{S^R}; y^{S^R})\}$ is a sequence of S-optimal allocations defined over a sequence of coalitions that contains all type 1^-'s and no type 1^+ that blocks the sequence $\{(\bar{x}^R_{\epsilon}; y^R)\}$ for large R.

■

Lemma 22. *Let $S^R \subset G^R$ be any coalition that includes all agents of type 1^- and no agents of type 1^+. Also, let $(\bar{x}^{S^R}; \bar{y}^{S^R})$ be an equal treatment Lindahl allocation for S^R let and $(\bar{x}^R; y^R)$ be an equal treatment Lindahl allocation for G^R. Then $\bar{y}^{S^R} < y^R$.*

Proof/

Suppose instead that $\bar{y}^{S^R} \geq y^R$. By lemma 11,

$$\forall \{r,t\} \in S^R \text{ and } \forall\, x, x' \geq 0, \text{ if } y^{S^R} \geq y^R,$$

$$\text{then } \forall\, (p^{R,r,t}; q^{R,r,t}) \in H^t(x, y^R) \text{ and } \forall\, (p^{S^R,r,t}; q^{S^R,r,t}) \in H^t(x', \bar{y}^{S^R}),$$

$$\frac{q^{R,r,t}}{p^{R,r,t}} \geq \frac{q^{S^R,r,t}}{p^{S^R,r,t}}.$$

Also by assumption, the agents of type 1^+, who are not included in S^R, are never satiated in the public good, $q^{R,1^+}/p^{R,1^+} > 0$. So,

$$\sum_{\{r,t\}\in G^R} \frac{q^{R,r,t}}{p^{R,r,t}} > \sum_{\{r,t\}\in S^R} \frac{q^{S^R,r,t}}{p^{S^R,r,t}}.$$

But there is only one public good, and the production set is a cone. Thus the marginal cost of public good is unique. Then let

$$(\tilde{p}; \tilde{q}) = MC\left(\sum_{\{r,t\}\in S^R} [x^{S^R,r,t} - \omega^t]; \bar{y}^{S^R}\right).$$

But since $(x^{S^R}; \bar{y}^{S^R})$ is S^R-optimal,

$$\sum_{\{r,t\}\in G^R} \frac{q^{R,r,t}}{p^{R,r,t}} > \sum_{\{r,t\}\in S^R} \frac{q^{S^R,r,t}}{p^{S^R,r,t}} = \frac{\tilde{q}}{\tilde{p}}.$$

Therefore, $(\bar{x}^R; y^R)$ does not satisfy the Samuelson conditions, and so could not be a Lindahl allocation.

■

Theorem 3. *For any sequence $\{\mathcal{E}^R\}$ of one private good, one public good economies satisfying assumptions A1, A2, A3, A7, A8, A9, A10, and B1-B3, the core does not converge to the set of Lindahl allocations.*

Proof/

First it will be shown that the conditions of lemma 21 must be false, and therefore there must exist an $\epsilon' > 0$ such that for any R' there is an $R \geq R'$ such that $(\bar{x}^R_{\epsilon'}; y^R) \in C(G^R)$. Suppose not. Then by lemma 21, there exists a sequence of coalitions $\{S^R\}$ that contain all agents of type 1^- and no agents of type 1^+, such that there exists R' such that for $R \geq R'$, S^R can block the allocation $(\bar{x}^R_\epsilon; y^R)$ with an S^R-optimal allocation $(x^{S^R}; y^{S^R})$ for which

$$\lim_{R\to\infty} x^{S^R,1^-} = \omega^1.$$

Recall that $(\bar{x}^{S^R}; \bar{y}^{S^R})$ is defined to be any equal treatment Lindahl allocation over the coalition S^R

1. Suppose first that $y^{S^R} > \bar{y}^{S^R}$. Then lemma 10

$$\forall\, (\bar{p}^{S^R,1}; \bar{q}^{S^R,1}) \in H^1(\bar{x}^{S^R,1}; \bar{y}^{S^R}),\ \text{and}\ (p^{S^R,1}; q^{S^R,1}) \in H^1(x^{S^R,1}; y^{S^R}),$$

$$\frac{\bar{q}^{S^R,1}}{\bar{p}^{S^R,1}} > \frac{q^{S^R,1}}{p^{S^R,1}}.$$

Also by lemma 11, for all $t \in \mathcal{T}$,

$$\forall\, (\bar{p}^{S^R,t}; \bar{q}^{S^R,t}) \in H^t(\bar{x}^{S^R,t}; \bar{y}^{S^R}),\ \text{and}\ (p^{S^R,t}; q^{S^R,t}) \in H^1(x^{S^R,t}; y^{S^R}),$$

$$\frac{\bar{q}^{S^R,t}}{\bar{p}^{S^R,t}} \geq \frac{q^{S^R,t}}{p^{S^R,t}}.$$

Again, since there is only one public good and the technology is conic, let

$$(\tilde{p};\tilde{q}) = MC\left(\sum_{\{r,t\}\in S^R}[x^{S^R,r,t}-\omega^t];\bar{y}^{S^R}\right).$$

But then

$$\sum_{\{r,t\}\in S^R}\frac{q^{S^R,t}}{p^{S^R,t}} < \sum_{\{r,t\}\in S^R}\frac{\bar{q}^{S^R,t}}{\bar{p}^{S^R,t}} = \frac{\tilde{q}}{\tilde{p}}.$$

That is to say, the Samuelson conditions are not satisfied at $(x^{S^R};y^{S^R})$. Thus, $(x^{S^R};y^{S^R})$ cannot be Pareto optimal, in contradiction of the hypothesis.

2. Now suppose that $y^{S^R} < \bar{y}^{S^R}$. Then by lemma 10,

$$\forall\ (\bar{p}^{S^R,1};\bar{q}^{S^R,1}) \in H^1(\bar{x}^{S^R,1};\bar{y}^{S^R}),\ \text{and}\ (p^{S^R,1};q^{S^R,1}) \in H^1(x^{S^R,1};y^{S^R}),$$

$$\frac{\bar{q}^{S^R,1}}{\bar{p}^{S^R,1}} < \frac{q^{S^R,1}}{p^{S^R,1}}.$$

Also by lemma 11, for all $t \in \mathcal{T}$,

$$\forall\ (\bar{p}^{S^R,t};\bar{q}^{S^R,t}) \in H^t(\bar{x}^{S^R,t};\bar{y}^{S^R}),\ \text{and}\ (p^{S^R,t};q^{S^R,t}) \in H^1(x^{S^R,t};y^{S^R}),$$

$$\frac{\bar{q}^{S^R,t}}{\bar{p}^{S^R,t}} \leq \frac{q^{S^R,t}}{p^{S^R,t}}.$$

But then

$$\sum_{\{r,t\}\in S^R}\frac{q^{S^R,t}}{p^{S^R,t}} > \sum_{\{r,t\}\in S^R}\frac{\bar{q}^{S^R,t}}{\bar{p}^{S^R,t}} = \frac{\tilde{q}}{\tilde{p}}.$$

Again the Samuelson conditions are not satisfied at $(x^{S^R};y^{S^R})$. Thus, $(x^{S^R};y^{S^R})$ cannot be Pareto optimal, in contradiction of the hypothesis.

3. Finally suppose that $y^{S^R} = \bar{y}^{S^R}$. But then since all agents not of type 1 are at least as well off at the blocking allocation as they are at the grand coalition's allocation, and by lemma 22 the grand coalition produces more public good at a Lindahl allocation as does the subcoalition S^R,

$$\forall\ \{r,t\} \in S^R \text{ s.t. } t \neq 1, x^{S^R,r,t} \geq \bar{x}^{S^R,r,t}.$$

Also, by lemma 21, $x^{S^R,1} \to \omega^1$. But by assumption 10,

$$x^{S^R,1} \to \omega^1 > \omega^1 - \epsilon \geq \bar{x}^{S^R,1}.$$

But then $\sum_{\{r,t\} \in S^R}(\omega^t - x^{S^R,r,t}) = y^{S^R} < (\omega^t - \bar{x}^{S^R,r,t}) = \bar{y}^{S^R}$. Contradicting the hypothesis of case 3.

Therefore there must exist an $\epsilon' > 0$ such that $(\bar{x}^R_{\epsilon'}; y^R)$ is in the core for economies of arbitrarily large size. If not then there must exist a coalition containing all the agents of type 1^- and no agents of type 1^- that blocks $(\bar{x}^R_{\epsilon}; y^R)$ with a Pareto optimal allocation that has neither more, nor less, nor exactly $\bar{y}^{S^R}$ public good. Clearly this is impossible.

It only remains to show that $(\bar{x}^R_{\epsilon'}; y^R)$ is not a Lindahl allocation. But this is immediate since by smoothness, the supporting price ratio is unique. Thus $OC^1(y^R)$ has a unique value $\bar{x}^{R,1}$. But then all agents of type 1^- are ϵ' below their offer curves, while agents of type 1^+ are ϵ' above their offer curves at this core allocation. Thus, for large R, $(\bar{x}^R_{\epsilon'}; y^R) \in C(G^R)$ and yet

$$NOC(\epsilon'; \bar{x}^R_{\epsilon'}; y^R) = \frac{1}{T}.$$

Therefore, the core does not converge to the set of Lindahl allocations.

■

5. Conclusion

Muench showed that core of a public goods continuum economy is not necessarily equivalent to the set of Lindahl allocations. The question of the generality of this result was left open. This paper goes part way to giving an answer. Both asymptotic satiation and strict non-satiation are shown to be sufficient to guarantee core convergence for a class of convex and monotone public goods economies. These two conditions, however, appear to describe be extreme cases. Asymptotic satiation implies that the offer correspondences converge to the endowment, and that in the limit, a negligible part of the private goods are devoted to public goods production at any core allocation. Strict non-satiation, on the other hand, implies that the offer curves converge to zero, and that essentially all of the private goods are devoted to public goods production at any core allocation in the limit. The last section of the paper show that for a more restricted class of economies, the core does not converge in the intermediate case where the offer curves are strictly bounded away from both the endowment point and zero.

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