The Ordinal Egalitarian Bargaining Solution for Finite Choice Sets †

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Abstract

Rubinstein, Safra and Thomson (1992) introduced the Ordinal Nash Bargaining Solution. They prove that Pareto optimality, ordinal invariance, ordinal symmetry, and IIA characterize this solution. A feature of their work is that attention is restricted to a domain of social choice problems with an infinite set of basic allocations. We introduce an alternative approach to solving finite social choice problems using a new notion called the Ordinal Egalitarian (OE) bargaining solution. This suggests the middle ranked allocation (or a lottery over the two middle ranked allocations) of the Pareto set as an outcome. We show that the OE solution is characterized by weak credible optimality, ordinal symmetry and independence of redundant alternatives. We conclude by arguing that what allows us to make progress on this problem is that with finite choice sets, the counting metric is a natural and fully ordinal way to measure gains and losses to agents seeking to solve bargaining problems.

Keywords: Bargaining Theory, Non-expected Utility Theory, Cooperative Games, Ordinal Preferences, Egalitarian Solution, Counting Metric.

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1. Introduction

Nash (1950) introduced the formal notion of a bargaining problem as an ordered pair (S, d) where S is interpreted as the set of feasible utility payoffs and d as the disagreement point agents receive if they fail to compromise on an outcome. Nash assumed that S is a convex set and justified this by imposing the hypothesis that agents' preferences satisfy the von-Neumann Morgenstern assumptions. A vast bargaining literature has since emerged, the great majority of which has adopted Nash's cardinal axiom of affine invariance of utilities (or scale invariance).

In almost every other area of theoretical economics, only the ordinal content of preferences is considered. Policy conclusions based on interpersonal comparisons of cardinal utility are rightly viewed with suspicion. It would therefore be extremely desirable to be able to characterize fair allocation procedures that similarly relied only on the ordinal information in agents' preferences. This is difficult, however, as bargaining theory is fundamentally concerned with balancing the welfare of agents. This strongly invites such interpersonal comparisons. In fact, Shapley (1969) provided a counterexample for a two person bargaining problem that showed there does not exist an ordinally invariant, efficient and strictly individually rational (and therefore symmetric) solution for a broad domain of problems. Interestingly, this counterexample does not seem to extend to the case of more than two players as shown by Shubik (1982). In fact, a class of such solutions has recently been characterized by Kibris (2004).

In showing this, Shapley followed Nash in interpreting the pair (S, d) as utility allocations. Thus, he implicitly used the welfarist axiom that any two problems with the same image in utility space should have the same solution when he established his impossibility theorem. This implicit axiom has been criticized in Roemer (1986), among others. As a result, Shapley's theorem does not logically exclude the existence of sensible (as defined by Shapley) ordinal bargaining solutions when the solution concepts and axioms are defined over the space of fundamental allocations instead of their image in utility space. [1]

[1] In rejecting the welfarist axiom, we are explicitly accepting the possibility that two problems that

In an ambitious paper, Rubinstein, Safra and Thomson (1992) (henceforth, RST) propose a more general approach. They adopt an abstract framework similar to Nash and propose an ordinal version of the Nash bargaining solution. They provide a characterization of this ordinal solution for a fairly general class of non-expected utility preferences. To obtain their results, RST impose the assumption that the feasible set is "convex". Convexity in this context is really a joint assumption on the nature of the feasible set and the class of admissible preferences. It is satisfied for example by "divide the dollar" problems when the utility functions are concave.

One property of RST's approach is that it requires that the feasible set have an infinite number of elements. It does not provide a solution for bargaining problems that have only a finite set of alternatives or even for problems derived as the set of lotteries over a finite set of basic alternatives. RST's approach has also been criticized by Grant and Kajii (1995) who demonstrate that RST's assumption of convexity jointly with the assumption of homogeneity of preferences implicitly induces a cardinal characterization of the ordinal Nash solution. This is a serious challenge to the motivation that underlies RST's approach.

Several other authors have worked on similar problems. Notable contributions include Dhillon and Mertens (1999) who propose a domain restriction that allows them to characterize a well defined ordinal utilitarian solution, Hanany and Safra (2000) who investigate the existence of the ordinal Nash solution, and Safra and Samet (2004) and Samet and Safra (2005) who construct a class of solutions for problems with more than two bargainers.

In this paper, we continue the important program of finding solutions to bargaining problems that do not rely on cardinal preferences. In light of Grant and Kajii's criticism, we abandon RST's convexity hypothesis. Since this was a joint assumption on allocations and preferences, there are two separate dimensions of this relaxation. First, we drop RST's assumptions that preferences are "quasiconcave" and substitute

have the same image in expected utility space may have different solutions if the underlying set of fundamental alternatives happens to differ. See also Sakovics (2004).

a requirement that they be "quasiconvex" instead. Basically, quasiconvexity requires that agents weakly prefer a lottery ℓ to a compound lottery of ℓ and ℓ' where $\ell \sim \ell'$. In a sense, this is saying that less uncertainty is better. We argue below that the experimental evidence supports quasiconvexity more than quasiconcavity. Second, we drop the requirement that the fundamental space of outcomes is convex and instead require that the set of alternatives is, in fact, finite.[2]

Finite bargaining sets seem to arise quite naturally in a number of contexts. Consider, for example, assignment or matching problems (people to jobs, professors to course offerings, families to houses, firms to broadcast or bandwidth licenses) or any problem in which there are natural increments in the allocations (auctions with minimum bid increments, parcels of land or lots of goods that must be allocated as whole units are good examples; even in the divide the dollar game, one cannot give agents fractions of pennies). Thus, problems with finite underlying feasible sets may even be more of the rule than the exception.

Given that many choice problems do involve a finite set of real alternatives, one has a choice of settling either on one of these alternatives or instead on one of the uncountably infinite lotteries over these basic alternatives. Both approaches have their merits. Settling on a non-random solution that satisfies an appealing set of axioms means that the outcome will satisfy the axioms both ex-ante and ex-post.[3] We treat the case in which lotteries are excluded in section 2, below. In contrast, if a lottery is proposed as a solution, the outcome will satisfy the axioms, ex-ante, but not ex-post after the lottery has been resolved and the agents take home their winnings. On the other hand, one has to acknowledge that lotteries are in fact feasible choices, and thus, perhaps, should not be a priori excluded as solution outcomes. This becomes complicated if preferences are purely ordinal and, in particular, do not necessarily satisfy Savage's independence axiom. We treat this more difficult case in section 3.

[2] See for example Mariotti (1998) who considers a domain of finite bargaining problems and offers a mulitvalued solution that retains a degree of cardinality.

[3] See Conley and Wilkie (1996) for additional discussion on this point.

Our main contribution is to define and characterize the Ordinal Egalitarian solution. Basically, this takes either the middle ranked point in the Pareto set, or the 50/50 lottery over the two middle ranked points as the solution to any finite bargaining problem. We show that when agents are ordinally risk averse (and preferences satisfy other standard domain restrictions), the OE solution is characterized by symmetry, independence of redundant alternatives and weak credible optimality. The ordinal risk aversion assumption is consistent with the "mixed fanning" result that Harless and Camerer (1994) found to have the best predictive properties in experimental tests of behavior under uncertainty. If we restrict the domain of preferences further to require that agents are ordinally risk neutral, then weak credible optimality can be replaced by weak Pareto optimality in the characterization.

2. The Ordinal Egalitarian Solution with Nonprobabilistic Outcomes

To illustrate our solution concept, we begin by considering a very simple class of problems in which we allow only deterministic outcomes. In the next section we generalize the domain and introduce lotteries. Each agent i has an ordinal preference ranking $\succeq_i$ over an abstract space of concrete alternatives which we denote $\mathcal{A}$. We assume that for all $i = 1, 2$ that $\succeq_i$ is complete and transitive and derive the strong preference and indifference relation from the weak relation in the ordinary way.

A social choice problem $S \subset \mathcal{A}$ is a selection from the basic set of social alternatives. In this paper we consider the class of two agent choice problems Σ which satisfy three properties.

1. For all $S \in \Sigma$, it holds that S contains a finite set alternatives.
2. If $S \in \Sigma$, for all $\hat{S}$ such that $\hat{S} \subseteq S$, it holds that $\hat{S} \in \Sigma$.
3. For all $x, y \in S$ such that $x \neq y$ and all $i = 1, 2$, either $x \succ_i y$ or $y \succ_i x$. (That is, preferences are strict.)

A social choice solution in this context is a single-valued map $f : \Sigma \to \mathcal{A}$ such that for all $S \in \Sigma$ it holds that $f(S) \in S$.

Note that we will not need to define a disagreement point in this paper. This is driven by the fact that we make purely ordinal comparisons of relative losses when defining our solution and so do not need to measure them with respect to a fixed alternative.

With no lotteries, the Pareto optimal set consists simply of those alternatives that are not Pareto dominated by other alternatives. We will use the superscript *nl* in this section to remind ourselves that our axioms are defined in a way that disregards lotteries.

$$PO^{nl}(S) \equiv \{\hat{x} \in S \mid \nexists\ x \in S \text{ s.t. } \forall\ i = 1, 2\ x \succ_i \hat{x}\}.$$

Pareto Optimality^{nl} (PO^{nl}): For all $S \in \Sigma$ it is the case that $f(S) \in PO^{nl}(S)$.

We will need several preliminaries before we define our notion of symmetry. Roughly, a choice set is symmetric if the good and bad alternatives for each agent are in some sense equal. To make this more precise, we need to know for any given choice set $S \in \Sigma$ and $x \in S$, how many alternatives exist in the S that are strongly preferred by each agent. Formally, we define the *Cardinality of the Preferred Set for agent i* as follows:

$$CPS_i(x, S) \equiv \{\ |\, T\, |\ \text{ where } y \in T \text{ if and only if } y \in S, \text{ and } y \succ_i x\}$$

where $|\, T\, |$ denotes the cardinality of the set T.

Next, we need to know the ordinal ranking of the Pareto set for each player. Given the domain restriction that preferences are strict, this ranking is unambiguous. Formally, the *rank* of a Pareto Optimal alternative $x \in PO^{nl}(S)$ for agent i is the following:

$$RANK_i(x, S) \equiv CPS_i(x, PO^{nl}(S)) + 1$$

In words, we will say a set is *Ordinally Symmetric* if for any r, the r^{th} most preferred alternative on the PO set of each of the agents has exactly the same number of alternatives in the feasible set that are strongly preferred. Obviously, this is a very strong hypothesis, and so few sets will be ordinally symmetric. In particular, it is always possible to destroy the ordinal symmetry of any set by adding a single point. Note, however, that any problem consisting only of a Pareto set (with no Pareto dominated points) is automatically ordinally symmetric. Formally:

Ordinal symmetry^{nl}: A problem $S \in \Sigma$ is ordinally symmetric^{nl} if for all $x, y \in PO^{nl}(S)$ such that $RANK_1(x, S) = RANK_2(y, S)$ it holds that $CPS_1(x, S) = CPS_2(y, S)$.

A solution concept satisfies the axiom of *Symmetry*^{nl} if the solution to every ordinally symmetric set is symmetric.

Symmetry^{nl} (SYM^{nl}): If $S \in \Sigma$ is is ordinally symmetric^{nl}, then $CPS_1(f(S), S) = CPS_2(f(S), S)$.

We will use a weaker version of symmetry in the next section. The final axiom we need is called *Invariance to Pareto Irrelevant Alternatives*^{nl}. It simply says that if two sets of social alternatives have the same PO sets, then the solutions should be the same. Note that this implies PO, so we will be able to drop this axiom in our characterization.

Invariance to Pareto Irrelevant Alternatives^{nl} (IPIA^{nl}): For all $S, S' \in \Sigma$ if $PO^{nl}(S) = PO^{nl}(S')$ then $f(S) = f(S')$.

We begin by defining our solution on a restricted domain in which the Pareto set has an odd number of elements. This makes the characterization extremely transparent. In the next section, we generalize this. Call this domain Σ^{odd}

$$\Sigma^{odd} \equiv \{S \in \Sigma \mid \ |PO(S)| \ \text{ is odd } \}.$$

Give that the Pareto set is odd, the *Ordinal Egalitarian* solution is defined as follows:

$$OE^{nl} \equiv \left\{ x \in PO(S) \text{ s.t. } \forall\, i = 1,2, RANK_i(x,S) = \frac{\mid PO(S) \mid}{2} + \tfrac{1}{2} \right\}$$

This solution concept can be motivated as an iterative veto solution (see Anbarci 1993 who shows the relationship between the non-cooperative Iterative Veto Solution and Area Monotone Solution described in Anbarci and Bigelow 1994). We can imagine agents agreeing that they will settle on a Pareto optimal outcome, and then deciding which one by iteratively vetoing their least favorite remaining Pareto alternative. This process continues until another round of vetoes would leave no alternatives left. This rule is in the class of Unanimity Compromise solutions studied in Brams and Kilgour (2001) and Kibris and Sertel (2007). In particular, because the solution is restricted to be a selection from the set of Pareto optimal and Individually Rational points it corresponds to the *Imputational Compromise Solution* studied in Kibris and Sertel (2007) which is related to the *Equal Length Solution* axiomatized in Thomson (1996).

The major difference between the papers above and the current work is that we focus on single valued instead of multivalued solution concepts. We do so in this section by restricting the domain to finite problems having an odd numbered Pareto sets which in turn allows us to give a very concise characterization.

One should especially note the relationship between the solution defined in this section with the one defined in Sakovics (2004). Both solutions are single-valued, and on the domain of problems with odd Pareto sets, choose the same outcome. Sakovics does not provide a characterization of his solution. In addition, the solutions are different on the more general domain defined in the next section. In particular, the Sakovics solution chooses the most preferred outcome of agent 1 if there are two middle points instead of the 50/50 lottery. Thus, the Sakovics solution in not symmetric in general. Of course it would be interesting to have a 50/50 lottery over which agent is to be favored when there are two middle points (as a referee suggests), and regain a kind of symmetry as a result. However, we are not sure how exactly one would capture this axiomatically. Sakovics also discusses extending this solution to social choice problems with infinite feasible sets. This is a very intriguing idea if one has a non-cardinal way to measure an infinite set of choices for the purposes of comparing them to one another (as one has with the counting metric).

While we think that the deterministic solution proposed in this section is interesting (especially since the outcome it suggests satisfies the proposed axioms both ex-ante and ex-post, as we discus in the introduction), the main contribution of the current paper is to extend this single-valued solution concept to more general finite domains. The challenge is to allow the use of lotteries both to define the Pareto set and as solution outcomes while providing a purely ordinal characterization of the resulting solution concept. We elaborate on this point in the next section.

We now show our characterization:

Theorem 1. *A solution on the domain Σ^{odd} satisfies SYM^{nl} and $IPIA^{nl}$ if and only if $f = OE^{nl}$.*

Proof/

We start by showing that the OE^{nl} satisfies the two axioms.

SYM^{nl}: Suppose that S is ordinally symmetric and so for all $x, y \in PO(S)$ such that $RANK_1(x,S) = RANK_2(y,S)$ it holds that $CPS_1(x,S) = CPS_2(y,S)$. The solution $x = OE^{nl}(S)$ is in $PO(S)$, and $RANK_1(x,S) = RANK_2(x,S) = \frac{|\,PO(S)\,|}{2} + \frac{1}{2}$. Thus, $CPS_1(x,S) = CPS_2(x,S)$.

$IPIA^{nl}$: Since OE^{nl} takes the middle point of the PO set as the solution, all problems with the same PO set must have the same solution. Thus, for all $S, S' \in \Sigma^{odd}$ such that $PO^{nl}(S) = PO^{nl}(S')$, it holds that $f(S) = f(S')$.

Next we show that if a solution satisfies the axioms, then it must be the ordinal egalitarian solution. Consider any $S \in \Sigma^{odd}$, and let $\hat{S} \equiv PO(S)$. Note that $\hat{S}$ is ordinally symmetric by construction and so SYM^{nl} implies that $CPS_1(f(\hat{S}), S) = CPS_2(f(\hat{S}), S)$. But since every point in $\hat{S}$ is PO, the only point for which this condition is satisfied is the middle ranked element of the Pareto set. Thus, $f(\hat{S}) = OE^{nl}(\hat{S})$. Since by construction, however, it is also the case that $PO^{nl}(S) = PO^{nl}(\hat{S})$, $OE^{nl}(S) = OE^{nl}(\hat{S})$ and by $IPIA^{nl}$ we conclude that $f(S) = f(\hat{S}) = OE^{nl}(\hat{S}) = OE^{nl}(S)$.

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We close this section by noting that if we drop SYM^{nl}, the dictatorial solution that selects person 1's most preferred point satisfies $IPIA^{nl}$. On the other hand, if we drop $IPIA^{nl}$, we can define a solution which takes the lowest Pareto ranked symmetric point, if this is unique, and the OE^{nl} solution otherwise. Thus, the two axioms are independent.

3. Extending the Ordinal Egalitarian Solution to Lotteries.

Extending this characterization to allow lotteries in a way that preserves its ordinallity turns out to be subtle and requires us to explore the approaches used in nonexpected utility theory and decision theory. In this section we lay out the preliminaries that will eventually allow us to characterize the ordinal egalitarian solution in the space of lotteries over finite choice sets. We will also discuss a number of related approaches in the literature. This will provide motivation for the specific domain and axioms we use in our own characterization. We conclude with our characterization of the ordinal egalitarian solution. We begin with some notation.

Denote the set of lotteries over the set of alternatives in a social choice problem as $L(S)$. A particular lottery is denoted $\hat{\ell} = (\hat{p}_1, \ldots, \hat{p}_k; \hat{z}_1, \ldots, \hat{z}_k) \in L(S)$ where $(\hat{p}_1, \ldots, \hat{p}_k) \gg 0$ is a strict probability mixture and for $j = 1, \ldots, k$, $\hat{z}_j \in S$. Where it will not cause confusion, we will sometimes write $\mu\ell + (1-\mu)\hat{\ell}$ to represent a compound lottery over two lotteries, $px + (1-p)z$ to represent the simple lottery between two certain alternatives, and x to denote the trivial lottery over a single point. It will also be useful to know the alternatives that form the support for a given lottery. We denote this as follows:

$$Supp(\ell) \equiv \{(z_1, \ldots, z_k) \subseteq S \text{ where } \ell = (p_1, \ldots, p_k; z_1, \ldots, z_k)\}.$$

A social choice problem is now a pair $(S, \succeq)$ where $S \subset \mathcal{A}$ and $\succeq \equiv (\succeq_1, \succeq_2)$ is a pair of preference relations each defined over $L(S)$. Given a domain of problems Σ, a social choice solution in this context is a single-valued map $F : \Sigma \to L(\mathcal{A})$ such that for all $S \in \Sigma$ it holds that $F(S, \succeq) \in L(S)$.

In order to characterize any solution over the space of lotteries, we will need to impose a few weak regularity conditions on the preferences of agents. Note, however, that these conditions do not force preferences to be cardinal. In particular, our assumptions are much weaker than those required for expected utility to hold.

We will assume that $\succeq_i$ is a complete and transitive preference relation over the space of all lotteries over all social choice problems in the domain for agent i. We will also require the following domain restrictions on these preferences.

Archimedean Axiom (AA): For all $i = 1, 2$, all $(S, \succeq) \in \Sigma$ and all $x, y, z \in S$ such that $x \succeq_i y \succeq_i z$ and $x \succ_i z$, there exists a unique $p \in [0, 1]$ such that $px + (1 - p)z \sim_i y$.

First Order Stochastic Dominance (FOSD): For all $i = 1, 2$, all $(S, \succ) \in \Sigma$, and any $z_1, z_2 \in S$ such that $z_1 \succeq_i z_2$, then for any $\ell \in L(S)$, if $1 \geq p > q \geq 0$, $\ell' = pz_1 + (1 - p_1)\ell$, and $\hat{\ell} = qz_2 + (1 - q)\ell$, then $\ell' \succeq_i \hat{\ell}$.

Note that since we assume that the fundamental points are strictly ordered, we use the strict preference ordering in our definition of FOSD.

RST impose the requirement that preferences over lotteries be quasiconcave. This assumption together with their convexity hypothesis ensured the existence and uniqueness of the Nash solution. However, quasiconcavity is unappealing as experimental evidence suggests that preferences are more likely be quasiconvex than quasiconcave (see the surveys of Camerer 1989 or Starmer 1992.) Moreover, as Grant and Kajii (1995) point out, the quasiconcavity hypothesis added to the other requirements that RST impose on preferences is quite restrictive. Together, these assumptions rule out much of the behavior that motivates nonexpected utility models. We will require instead that preferences satisfy:

Quasiconvexity (QC): For all $i = 1, 2$, all $(S, \succeq) \in \Sigma$ and all $\ell, \hat{\ell} \in L(S)$, if $\ell \succeq_i \hat{\ell}$, then for all $\mu \in [0, 1]$ it holds that $\ell \succeq_i \mu\ell + (1 - \mu)\hat{\ell}$.

Let Σ_0 denote this base class of problems for which preferences satisfy the AA, FOSD, and QC axioms.

The intersection of quasi-concave and quasi-convex preferences are those that satisfy the betweenness property, see Fishburn (1983), Dekel (1986), Chew (1989) and Karni and Schmiedler (1991).

> Betweenness (B): For all $i = 1, 2$, all $(S, \succeq) \in \Sigma$ and all $\ell, \hat{\ell} \in L(S)$, if $\ell \sim_i \hat{\ell}$, then for all $\mu \in [0, 1]$ it holds that $\ell \sim_i \mu\ell + (1-\mu)\hat{\ell}$.

Let $\Sigma_B \subset \Sigma_0$ denote the class of bargaining problems where preferences satisfy the betweenness axiom.

RST also impose an axiom called *Conditional Substitution of Certainty Equivalents* (CCE) and a weakening called CCE*. Karni and Schmiedler (1991) refer to CCE* as the axiom *Substitution of Certainty Equivalents.* We will also use an axiom similar to CCE.[4]

> Ordinal Risk Aversion (ORA): For all $i = 1, 2$, all $(S, \succeq) \in \Sigma$, all $x \in S$ and $y \in S$, let $\ell = (p_1, p_2, z_1, z_2) \in L(S)$, and $\ell' = (q_1, q_2, z_1, z_2)$ be such that $x \sim_i \ell$ and $y \sim_i \ell'$. Then for all $\mu \in [0, 1]$ it holds that $\mu x + (1-\mu)y \succeq_i \mu\ell + (1-\mu)\ell'_i$.

Let $\Sigma_A \subset \Sigma_0$ denote the class of problems that satisfy the ORA axiom.

The ORA axiom can be interpreted as an aversion to mean preserving spreads.[5] In particular, it states that if we replace the lottery components of a compound lottery with their certainty equivalents then this simple lottery is weakly preferred. Note that the simple lottery has, in terms of the ordinal preference ranking, a range that is contained in the range of the compound lottery. Of course, the motivation for assuming

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[4] We remark that ORA implies the following weak version of QC: $\forall (S, \succeq) \in \Sigma$ and $\forall\ x \in S$, if $x \sim_i \hat{\ell}$, then $\forall \mu \in [0, 1]$ it holds that $x \succeq_i \mu x + (1-\mu)\hat{\ell}$.

[5] Rothchild and Stigliz (1970) show that cardinal risk aversion is equivalent to a dislike of mean preserving spreads. Following this tradition, we describe a dislike of *median* preserving spreads as ordinal risk aversion. Note that the standard expected utility "independence" axiom implies that ORA holds (in fact, it would imply $\mu x + (1-\mu)y \sim_i \mu\ell + (1-\mu)\ell'_i$).

that agents are risk averse is long established in the economics literature and will not be repeated here.

Note that the ORA axiom only requires that lotteries over two certain points are at least as good as the same lottery over certainty equivalent lotteries whereas the CCE axiom requires that lotteries over two certain points are at least as good as the same lottery over one of those points and a lottery equivalent to the other. Thus, there is no formal relation between CCE and ORA. Note that ORA also has similarities to the reduction axiom used in Segal (1990).

It is important to know that the domain of preferences we study is not empty. Consider the "Machina (1984, 1987) triangle" case of lotteries over three alternatives, a, b, c with $a \succeq b \succeq c$ (See Figure 1). The lottery l^* is the simple mixture over a, c that is indifferent to the degenerate lottery b. Notice that preferences exhibit the "fanning out" characteristic as we move from b toward a (these "start" from x), but the "fanning in" property as we move from b toward c (these "start" from y). Interestingly this is exactly the behavior that is most consistent with the experimental data, see for example Harless and Camerer (1994) or Starmer (1992). Based on this, the following provides an example of a class of nonexpected utility preferences that satisfy the ORA and QC axioms (and of course, AA and FOSD). Let $S = \{a, b, c\}$ and suppose that $a \succeq b \succeq c$. Let $\ell = (p_a, p_b, p_c, a, b, c)$. Now consider preferences with the following utility representation $U(\ell) = \sum_{z \in S} f_z(\ell) u_z$ where $f_c(\ell) = p_c, f_b(\ell) = p_b, f_a(\ell) = (p_a)^2/(1 - p_b)$,and $u_c = 0, u_a = 1, u_b = \alpha^2$ where $b \sim (\alpha, 1 - \alpha; a, c)$. The interpretation of these preferences is as follows: any compound lottery can be thought of as a compound lottery between the degenerate lottery b and a lottery between a and c. Therefore, an agent will get either the median outcome b, or face a lottery between the good outcome, a, and the bad outcome, c. Conditional on not getting b the decision maker "discounts" the good outcome, a. It is straightforward to verify that these preferences satisfy the axioms.

Figure 1 about here

In the following, we also sometimes make use of the stronger axiom of Ordinal Risk Neutrality.

Ordinal Risk Neutrality (ORN): For all $i = 1, 2$, all $(S, \succeq) \in \Sigma$, all $x \in S$ and $y \in S$, let $\ell = (p_1, p_2, z_1, z_2) \in L(S)$, and $\ell' = (q_1, q_2, z_1, z_2)$ be such that $x \sim_i \ell$ and $y \sim_i \ell'$. Then for all $\mu \in [0, 1]$, it holds that $\mu x + (1 - \mu) y \sim_i \mu\ell + (1 - \mu)\ell'_i$.

Let $\Sigma_N \subset \Sigma_A \subset \Sigma_0$ denote the class of problems that satisfy the ORN axiom.

At last we are ready to extend the axioms that characterize our solution to permit the use of lotteries. Of course, this implies that the weak Pareto set is now a set of lotteries:

$$WPO(S, \succeq) \equiv \{\hat{\ell} \in L(S) \mid \nexists\, \ell \in L(S) \text{ s.t. } \forall\, i = 1, 2,\ \ell \succ_i \hat{\ell}\}.$$

Weak Pareto Optimality (WPO): for all $(S, \succeq) \in \Sigma$, it is the case that $F(S, \succeq) \in WPO(S, \succeq)$.

It will be useful to know the set of basic alternatives that form the *Support of the Pareto optimal Set of lotteries*. We denote this as follows:

$$SPS(S, \succeq) \equiv \{x \in S \mid \exists\, \hat{\ell} \in WPO(S, \succeq) \text{ and } x \in Supp(\hat{\ell})\}.$$

It will also be useful to know the set of basic alternatives (degenerate lotteries) that are Pareto Optimal. Call this the *Degenerate Pareto Optimal set*.[6]

$$DPO(S, \succeq) \equiv WPO(S, \succeq) \bigcap S.$$

To define symmetry in the context of a finite ordinal social choice problem, we begin with the notion of a symmetric permutation operator.

[6] Please note the remark given after WCO is defined below.

Symmetric permutation operator (SPO): A one-to-one mapping of a social choice problem into itself, $\phi : S \to S$ is a symmetric permutation operator if for all $x, y \in S$ it holds that $y = \phi(x)$ if and only if $x = \phi(y)$ and for all $x, y \in S$ and $i \neq j$, $x \succeq_i y$ if and only if $\phi(x) \succeq_j \phi(y)$. [7]

A lottery $\hat{\ell}$ is said to be the *symmetric permutation* of the lottery ℓ under an SPO ϕ if for all elements x in the support of ℓ there exists $\hat{x} = \phi(x)$ in the support of $\hat{\ell}$ and in addition $\hat{p} = p$. A lottery ℓ is said to be a *fixed point* under the SPO ϕ if $\phi(\ell) = \ell$.

Ordinally Symmetric Problem: A problem $(S, \succeq) \in \Sigma$ is ordinally symmetric if there exists an SPO ϕ such that for all x, y if $y = \phi(x)$ then $\ell \succeq_1$ $(resp. \succ_2)$ x if and only if $\phi(\ell)) \succeq_2$ $(resp. \phi(\ell)) \succ_2 y)y$.[8]

Given this, following Grant and Kajii (1995), we define our *Symmetry* axiom as follows:

> Symmetry (SYM): For all $S \in \Sigma$ such that S is ordinally symmetric with respect to a symmetric permutation operator ϕ, it holds that $F(S, \succeq)$ is a fixed point of ϕ.

Note that we could also characterize our solution using a generalized version of the symmetry axiom used in the previous section (which excluded lotteries). Specifically, we could require that if a problem is ordinally symmetric in the sense used in section 2, then the solution must be a symmetric lottery (that is a lottery over ordinally symmetric points). We think the Grant and Kajii axiom is more natural, however, and prefer to stick to existing axioms as much as possible.

Next, we give an axiom in the spirit of invariance to Pareto irrelevant alternatives called *Independence of Redundant Alternatives.* The axiom is adapted from Dhillon and Mertens (1999) who use it in their characterization of relative utilitarianism. The

[7] It is not hard to show that the SPO that respects the preference ordering of the two agents is unique.

[8] Interested readers may contact the authors for a working paper version of this manuscript that includes a discussion of alternative definitions of symmetry.

intuition for the axiom is welfarist in nature. It suggests that if an alternative that was not chosen is no longer available but a perfect substitute lottery remains available, then the solution should not change.

Independence of Redundant Alternatives (IRA): Consider any pair of social choice problems $(S, \succeq), (S', \succeq') \in \Sigma$ such that $S' \subset S$, and let $\succeq'$ be the restriction of $\succeq$ to S'. If $DPO(S, \succeq) = DPO(S', \succeq')$ then $F(S, \succeq) = F(S', \succeq')$.

Next we propose the following definition of optimality:

$$WCO(S, \succeq) \equiv \{\ell \in L(S) \mid (i)\ Supp(\ell) \subseteq WPO(S, \succeq), \text{ and}$$
$$(ii) \text{ if } \exists\ \ell' \in L(S) \text{ s.t. } \ell' \succeq_i \ell \text{ for } i = 1, 2 \text{ and}$$
$$\ell' \succ_i \ell \text{ for some } i, \text{ then } \ \exists\ x \in Supp(\ell'), \text{ s.t. } x \notin WPO(S, \succeq)\}.$$

We say that a solution satisfies *Weak Credible Optimality* if it selects an outcome from this set:

Weak Credible Optimality (WCO): For all $(S, \succeq) \in \Sigma$, $F(S, \succeq) \in WCO(S, \succeq)$.

If $\ell \notin WCO(S, \succeq)$ then we will say that ℓ is *credibly dominated by some* ℓ'.

We conclude this section with two brief remarks:

Remark 1: Note that since any WPO point in S is undominated, it is also not credibly dominated. Thus, we could just as well have defined DPO, above, as the intersection of $WCO(S, \succeq)$ and S without changing the composition of the set. This is important as it means that the definition of the ordinal egalitarian solution and the axiom IRA used in its characterization is consistent with either view of optimality.

Remark 2: Also note that if preferences satisfy the von-Neumann Morgenstern independence axiom then F satisfies WCO if and only if it satisfies ex-ante Pareto optimality. Interested readers may ask the authors for an extended version of this paper in which this claim is proved.

Our objective is to provide a solution on our domain which is egalitarian in nature. The most natural thing to do in this spirit is to take either the middle ranked point in the Pareto set (as in section 2, above) or if the Pareto set is even, the 50/50 lottery over the two middle ranked points. Formally:

$$OE(S,\succeq) \equiv \begin{cases} x \in DPO(S,\succeq) \text{ s.t. } \forall\, i = 1,2, & \\ RANK_i(x,S) = \frac{\mid DPO(S,\succeq) \mid}{2} + \frac{1}{2} & \text{if } \mid DPO(S,\succeq) \mid \text{ is odd} \\ & \\ \ell^* = (\frac{1}{2},\frac{1}{2},x,y) \in DPO(S,\succeq) \text{ where} & \\ RANK_1(x,S) = RANK_2(y,S) = \frac{\mid DPO(S,\succeq) \mid}{2} & \text{if } \mid DPO(S,\succeq) \mid \text{ is even} \end{cases}$$

With these preliminaries we now provide two characterization theorems. Theorem 2 applies to the domain of problems in which agents' preferences satisfy ordinal risk aversion. For this domain, symmetry, weak credible optimality and independence of redundant alternatives characterize the ordinal egalitarian solution. We have relegated the proofs to the appendix.

Theorem 2. *A solution on Σ_A satisfies SYM, WCO and IRA if and only if $F = OE$.*

Proof/

See appendix

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Finally, if we are willing to further restrict the domain of preferences to those satisfying ordinal risk neutrality, the characterization can be strengthened to include full WPO instead of WCO.

Theorem 3. *A solution on Σ_N satisfies SYM, WPO and IRA if and only if $F = OE$.*

Proof/

See appendix.

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4. Conclusion

It is probably not surprising that the great majority of the literature on axiomatic bargaining theory relies to some extent on cardinal foundations. This is because the fundamental problem is to propose a balancing of welfare gains and/or losses over agents. Doing so without making at least implicit interpersonal comparisons of utility over agents is therefore very difficult.

This paper suggests that this problem is compounded by the insistence on including convex and therefore infinite choice sets in the domain of problems. It is extremely unclear how to compare the relative losses to agents when each is compromising by giving up an infinite (usually, uncountably infinite) number of preferred alternatives. Infinities (of the same order) are all equivalent and so one must use other more subjective metrics to compare gains or losses from any given compromise point.

When the choice set is finite, however, there is a very natural and fully cardinal metric available: the counting metric. We can simply count the number of preferred alternatives that each agent gives up to reach a compromise. As we argued in the body of the paper, finite underlying choice sets are at least as natural and perhaps more natural that infinite ones. Many real world problems are fundamentally discrete (who should I marry, what job should I take, where should I live?) and even those we commonly approximate as continuous may have some degree of granularity as a matter of practice (almost any division of wealth problem is subject to a minimum currency unit constraint). Of course, lotteries do introduce a kind of continuity in the outcome space; however, as long as the underlying choice is finite, the counting metric is still available to us.

We define and characterize the most obvious bargaining solution suggested by the counting metric: equal ordinal sacrifice of preferred allocations. More formally, the ordinal egalitarian solution is the middle ranked point in the Pareto set, if it exists, and the 50/50 lottery over the two middle ranked points if it does not. We show that if preferences satisfy ordinal risk aversion, the OE solution is characterized by weak credible optimality, ordinal symmetry and independence of redundant alternatives. We also show that if we strengthen this to an assumption that preferences satisfy ordinal risk neutrality, the OE solution is characterized by weak Pareto optimality, ordinal symmetry and independence of redundant alternatives.

Appendix

We begin with some preliminary results. The following lemma shows that if a lottery is Pareto dominated by another, there must also exist a simple lottery over exactly two alternatives that is Pareto dominant.

Lemma 1. *On Σ_0 let $\ell \notin WPO(L(S), \succeq)$ then there exists a lottery ℓ' where $\mid supp(\ell') \mid \leq 2$ which Pareto dominates ℓ.*

Proof/

We want to show that if a lottery ℓ is weakly Pareto dominated by any lottery, there exists another lottery with at most two allocations in its support that will also dominate ℓ. To see this, suppose that ℓ is dominated by a lottery $\bar{\ell} = (\bar{p}_1, \ldots, \bar{p}_n, \bar{z}_1, \ldots, \bar{z}_n)$ where $n > 2$. Note that $(\bar{p}_1, \ldots, \bar{p}_n)$ is in the relative interior of the $n-1$ dimensional simplex. Define the set of weakly inferior probability mixtures over $(\bar{z}_1, \ldots, \bar{z}_n)$ under agent 1's preferences as:

$$W_1(\bar{\ell}, \succeq) \equiv \{(\hat{p}_1, \ldots, \hat{p}_n) \in \Delta^{n-1} \mid \bar{\ell} \succeq_1 \hat{\ell}\}.$$

Note that by QC and FOSD, $W_1(\bar{\ell}, \succeq)$ is convex.

Recall that by assumption, S is strictly ranked by both agents. Let $(\bar{z}_1, \ldots, \bar{z}_n)$ be an ordered list of elements in the support of lottery $\bar{\ell}$ by agent 1 where z_n is the least preferred point. Then by FOSD $\mu \bar{z}_n + (1-\mu)\bar{\ell} \in W_1(\bar{\ell}, \succeq)$ for $\mu \in [0, 1]$ Thus, $W_1(\bar{\ell}, \succeq)$ has a nonempty interior.

It follows that there exists a hyperplane $H(p, \alpha)$ that supports $W(\bar{\ell}, \succeq_1)$ at $(\bar{p}_1, \ldots \bar{p}_n)$. By construction, if $(\tilde{p}_1, \ldots, \tilde{p}_n) \in H(p_1, \ldots p_n, \alpha)$, then $\tilde{\ell} \succeq_1 \bar{\ell}$. Since $(p_1, \ldots, p_n, \alpha) \cap \Delta^{n-1}$ is a polyhedron, any point in this intersection can expressed as a weighted sum of the polyhedron's extreme points. Note that these points lie on the boundary of the simplex and are therefore lotteries over at most $n-1$ allocations. Also note that they can be weakly ranked by $\succeq_2$. Let this ranking be $\ell^1, \ell^2, \ldots, \ell^k$. By QC and FOSD, we know that $\ell^1 \succeq_2 \bar{\ell}$. However, since $H(p_1, \ldots, p_n, \alpha)$ supports $W_1(\bar{\ell}, \succeq)$ and $\ell^1 \in H(p_1, \ldots, p_n, \alpha)$, $\ell^1 \succeq_1 \bar{\ell}$. This implies that there exists a lottery supported by at most $n-1$ points which is at least as good to both agents as $\bar{\ell}$ and which therefore Pareto dominates ℓ. Since the same argument can be applied for all $n > 2$, we conclude that there also exists a lottery with at most two allocations in its support that Pareto dominates ℓ.

■

We use this to prove our next series of lemmas which demonstrate that our solution satisfies the axioms. We begin with the axiom of Weak Credible Optimality.

Lemma 2. *On Σ_A, the OE solution satisfies WCO.*

Proof/

Let $(S, \succeq) \in \Sigma_A$ be given. First observe that by construction if $x \in supp(OE(S, \succeq))$ then $x \in DPO(S, \succeq)$ and so condition (i) of WCO is satisfied. To check the second condition we consider two cases.

(1) If the cardinality of the set $DPO(S, \succeq)$ is odd, then by definition $OE(S, \succeq)$ is a nonrandom allocation and $OE(S, \succeq) = x \in WPO(S, \succeq) \subseteq WCO(S, \succeq)$.

(2) If the cardinality of $DPO(S, \succeq)$ is even, then $OE(S, \succeq) = \ell^* = (\frac{1}{2}, \frac{1}{2}, x, y)$, where $RANK_1(x, S) = RANK_2(y, S) = \frac{|\,SPS(S)\,|}{2}$. If $\ell^* \notin WCO(S, \succeq)$, however, then there must exist a Pareto optimal lottery $\bar{\ell}$ which dominates it with $Supp(\bar{\ell}) \subseteq WPO(S, \succeq)$. Without loss of generality, we assume that agent 1 strictly prefers the lottery $\bar{\ell}$ in the following. We consider two subcases.

a. Suppose first that $\bar{\ell}$ is a degenerate lottery that places all weight on some allocation z. We also know that z is Pareto optimal and preferred by both agents to ℓ^*. Thus, for both agents, it cannot be the case that z is inferior to both x or y at the same time as this would violate FOSD. Suppose that one of the following ranking holds: $x \succeq_1 z \succeq_1 y$, and $y \succeq_2 z \succeq_2 x$, or the reverse. This, however, would violate the hypothesis that ℓ^* is the OE solution since z would then be the middle ranked point by both agents and so would be the OE solution instead.
The only other possibility is the case that z is preferred to both x and y under *both* agents' preferences. Then z Pareto dominates x and y which contradicts the hypothesis that they are elements of $WPO(S, \succeq)$.

b. Now suppose that $\bar{\ell}$ is a nontrivial lottery. By Lemma 1 we can assume without loss of generality that this has exactly two elements in its support, $\bar{\ell} = (\lambda, 1-\lambda; z, w)$, which by Lemma 1 are both Pareto optimal. We also know that points in the support of $\bar{\ell}$ must be inversely ranked by the agents (since they are in the Pareto set). Since x and y are the support of the OE solution they must be in the middle of each agent's ranking of the Pareto sets. As a result, FOSD allows us to conclude that $z \succ_1 x \succ_1 \bar{\ell} \succ_1 y \succ_1 w$ and $w \succ_2 y \succ_2 \bar{\ell} \succ_2 x \succ_2 z$.
Now, by AA, we know that the following lotteries are well defined:

$$\ell_1^x = (\mu_1, 1-\mu_2; z, w) \sim_1 x$$

$$\ell_1^y = (\nu_1, 1-\nu_1; z, w) \sim_1 y$$

$$\ell_2^x = (\mu_2, 1-\mu_2; z, w) \sim_2 x$$

$$\ell_2^y = (\nu_2, 1-\nu_2; z, w) \sim_2 y.$$

We claim that $\mu_1 \geq \mu_2$ and $\nu_1 \geq \nu_2$. Consider the first inequality and suppose instead that $\mu_1 < \mu_2$. By FOSD applied to agent 2, it is immediate that $\ell_1^x \succ_2 \ell_2^x$.

This would imply that ℓ_1^x weakly Pareto dominates x, contradicting the hypothesis x is Pareto optimal. A symmetric argument applies to the second inequality. Note that this implies:

$$\tfrac{1}{2}(\mu_2 + \nu_2) \leq \tfrac{1}{2}(\mu_1 + \nu_1).$$

By ORA applied to player 1, we know that $(\frac{1}{2}, \frac{1}{2}, x, y) = \ell^* \succeq_1 (\frac{1}{2}, \frac{1}{2}, \ell_1^x, \ell_1^y)$. By hypothesis, $\bar{\ell} \succ_1 \ell^*$, so by FOSD we have that $\lambda > \frac{1}{2}(\mu_1 + \nu_1)$. By ORA applied to player 2 we have that $(\frac{1}{2}, \frac{1}{2}; x, y) = \ell^* \succeq_2 (\frac{1}{2}, \frac{1}{2}; \ell_2^x, \ell_2^y)$. By hypothesis, $\bar{\ell} \succeq_2 \ell^*$, and so by FOSD we have that $\lambda \leq \frac{1}{2}(\mu_2 + \nu_2)$. But then:

$$\tfrac{1}{2}(\mu_2 + \nu_2) > \tfrac{1}{2}(\mu_1 + \nu_1),$$

a contradiction. Thus, ℓ^* is not dominated by any such simple lottery over two points and so $OE(S, \succeq) \in WCO(S, \succeq)$. This completes the proof.

■

We can use essentially the same argument to show that if preferences satisfy ordinal risk neutrality instead of ordinal risk aversion then the OE solution satisfies full WPO.

Corollary 1. *On Σ_N, the OE solution satisfies WPO.*

Proof/

Let $(S, \succeq) \in \Sigma_N$ be given and $\ell^* = OE(S, \succeq)$. As $\Sigma_N \subset \Sigma_A$, by Lemma 2 $\ell^* \in DCO(S, \succeq)$ and $supp(\ell^*) \subset WPO(S, \succeq)$. Moreover, following the proof of Lemma 2, the only case in which the OE solution is not WPO is when $OE(S, \succeq) = \ell^* = (1/2, 1/2; x, y)$ and there is a candidate lottery $\ell' = (\mu, 1 - \mu; w, z)$ that could possibly Pareto dominate ℓ^* where(i) $w, z \notin DPO(S, \succeq)$ and (ii)$x \succeq_1 w \succeq_1 z \succeq_1 y$, with the inverse for agent 2. As preferences satisfy FOSD, if $\ell' \in WPO(S, \succeq)$ then neither w nor z are Pareto dominated by any point in S. Moreover, if w or z is Pareto dominated by lottery, $\ell'' = (\phi, 1 - \phi; a, b)$, since by hypothesis ℓ' Pareto dominates ℓ^*, it follows that ℓ'' Pareto dominates ℓ^*. However from the proof of Lemma 2, it cannot be the case that $a \succeq_1 x, \succeq_1 y, \succeq_1 b$. Thus on Σ_N, if $w, z \notin DPO(S, \succeq)$ then w and z must be Pareto dominated by some lotteries over the Pareto Optimal points x and y. Then using the argument in part 2 of the proof of Lemma 2, by the Archimedean Axiom we may define the mixtures over x and y such that w and z are certainty equivalents of these mixtures:

$$\begin{aligned}
\ell_1^w &= (\mu_1, 1 - \mu_2; x, y) \sim_1 w \\
\ell_1^z &= (\nu_1, 1 - \nu_1; x, y) \sim_1 z \\
\ell_2^w &= (\mu_2, 1 - \mu_2; x, y) \sim_2 w \\
\ell_2^z &= (\nu_2, 1 - \nu_2; x, y) \sim_2 z.
\end{aligned}$$

Therefore we can adapt the argument of the proof in case 2 in Lemma 2, by replacing weak preference with indifference, to establish that if $w, z \notin DPO(S, \succeq)$, then ℓ' cannot Pareto dominate ℓ^*. Thus $\ell^* \in WPO(S, \succeq)$.

■

Next we show that given ORA, the OE solution satisfies IRA.

Lemma 3. *On Σ_A, the OE solution satisfies IRA.*

Proof/

This is immediate from the definition of OE. Let $(S, \succeq) \in \Sigma_A$ be given then $OE(S, \succeq)$ is either the middle ranked point of $DPO(S, \succeq)$, x if the cardinality of $DPO(S, \succeq)$ is odd, or the lottery $(\frac{1}{2}, \frac{1}{2}; x, y)$ where x and y are the two middle ranked points in $DPO(S, \succeq)$ when the cardinality of $DPO(S, \succeq)$ is even. Let $(S' \succeq')$ be such that the hypothesis of IRA applies to the pair $DPO(S, \succeq)$, $DPO(S', \succeq')$. Then by definition if the cardinality of $DPO(S, \succeq)$ is odd then x is the middle ranked point of $DPO(S', \succeq')$ and so $x = OE(S', \succeq')$. The analogous argument holds is the cardinality of $DPO(S, \succeq)$ is even. Thus $OE(S, \succeq) = OE(S', \succeq')$.

■

Finally, we show the OE solution satisfies Symmetry.

Lemma 4. *On Σ_A, the OE solution satisfies SYM.*

Proof/

Take any ordinally symmetric problem $(S, \succeq) \in \Sigma_A$ with associated symmetric permutation operator ϕ. We must show that $OE(S, \succeq)$ is a fixed point of ϕ.

1. Suppose that the cardinality of $DPO(S, \succeq)$ is odd. Then $OE(S, \succeq) = x$ is the middle ranked point in the DPO set. Suppose that $\phi(x) = y \neq x$. We first show that if x is Pareto optimal then $\phi(x)$ must be Pareto optimal. Suppose not, then there exists some lottery ℓ, such that (without loss of generality) $\ell \succ_1 y$ and $\ell \succeq_2 y$. Then by the definition of symmetry for $\ell' = \phi(\ell)$ it holds that $\ell' \succ_2 x$ and $\ell' \succeq_1 x$. But this contradicts the hypothesis that $x \in WPO(S, \succeq)$. Therefore, $y \in DPO(S, \succeq)$.
   Now define $A = \{z \in DPO(S, \succeq) \mid z \succeq_1 x\}$ and $B = \{z \in DPO(S, \succeq) \mid z \succeq_2 y\}$ Suppose without loss of generality that $x \succ_1 y$. Then as $x, y \in DPO(S \succeq)$, it must be that $y \succ_2 x$. Note that if $z \in A$, then by SYM, $\phi(z) \succ_2 \phi(x) = y$. By the argument above, $\phi(z) \in DPO(S, \succeq)$. Therefore, $z \in A$ if and only if $\phi(z) \in B$. It follows that $|A| = |B|$. Therefore, if $x \neq y$, then x could not be the middle ranked point in the Pareto set, which contradicts the hypothesis that $x = OE(S, \succeq)$.
2. Suppose now that the cardinality of $DPO(S, \succeq)$ is even. Then $OE(S, \succeq) = (\frac{1}{2}, \frac{1}{2}; x, y)$ suppose now that $\phi(x) = z \neq y$, then we may replicate the previous argument using x and z to obtain a similar contradiction.

■

Having shown that the OE solution satisfies our axioms, we need one additional technical lemma to complete our characterizations.

Lemma 5. *Let $(S, \succeq) \in \Sigma_0$. Define $S' = DPO(S \succeq)$ and let $\succeq'$ be the restriction of $\succeq$ on S', then $WCO(S', \succeq') = WPO(S', \succeq') = WCO(S, \succeq)$.*

Proof/

First, observe by definition that for all $x \in S'$, it must be that $x \in WPO(S, \succeq)$ and since $S' \subset S$, it also follows that $x \in WPO(S' \succeq')$. Suppose that $\ell \in L(S')$ Pareto dominates some $\ell' \in L(S')$. Then as the support of ℓ is Pareto optimal in $L(S')$, it must be that ℓ credibly dominates ℓ' as well. Therefore, we have that $WCO(S', \succeq') \subset WPO(S', \succeq')$. Also, since for all $x \in S'$, $x \in WPO(S, \succeq)$, if $\ell \in WPO(S' \succeq')$ then the support of ℓ is Pareto optimal and so $WPO(S', \succeq') \subset WCO(S' \succeq')$. Thus, $WCO(S', \succeq') = WPO(S', \succeq')$.

We now prove that $WPO(S', \succeq') = WCO(S, \succeq)$.

(i) Let $\ell \in WPO(S', \succeq')$. By definition:

(a) *supp*$(\ell) \subset WPO(S, \succeq)$ and
(b) $\nexists\, \ell' \in L(S')$ such that ℓ' Pareto dominates ℓ.

But then as $S' = S \cap WPO(S, \succeq) \equiv DPO(S', \succeq')$, there does not exist an $\ell'' \in L(S)$ that credibly dominates ℓ. Thus $WPO(S', \succeq') \subset WCO(S, \succeq)$.

(ii) Let $\ell \in WCO(S, \succeq)$. By the definition of $WCO(S, \succeq)$ we have that *supp*$(\ell) \subset DPO(S, \succeq)$ and so $\ell \in L(S')$. We now consider two cases. First, suppose $\ell \in WPO(S, \succeq)$. Then $\ell \in WPO(S', \succeq')$, and as *supp*$(\ell) \subset WPO(S', \succeq')$, we conclude that $\ell \in WCO(S', \succeq')$. Second, suppose instead that $\ell \notin WPO(S, \succeq)$. Then as $\ell \in WCO(S, \succeq)$, it must be Pareto dominated by a lottery ℓ' with an element y in its support that is Pareto dominated in $L(S)$. But then $y \notin S'$, and thus $\ell' \notin L(S')$. Thus, ℓ is not Pareto dominated in $L(S')$. We conclude that $WCO(S, \succeq) \subset WPO(S', \succeq')$.

Therefore: $WCO(S', \succeq') = WPO(S', \succeq') = WCO(S, \succeq)$.

■

With these preliminaries we now provide our characterization theorems. Theorem 2 applies to the domain of problems in which agents' preferences satisfy ordinal risk aversion. For this domain, symmetry, weak credible optimality and independence of redundant alternatives characterize the ordinal egalitarian solution.

Theorem 2. *A solution on Σ_A satisfies SYM, WCO and IRA if and only if $F = OE$.*

Proof/

By Lemmas 4, 2 and 3, the OE solution satisfies SYM, WCO and IRA.

To show the converse, consider any $(S, \succeq) \in \Sigma_A$, and let $S' \equiv DPO(S, \succeq)$. Define $(S', \succeq')$ where $\succeq'$ is the restriction of $\succeq$ on S'. By construction, if $x \in S'$, then x is Pareto optimal with respect to $\succeq$. By Lemma 5, $WCO(S', \succeq') = WCO(S, \succeq)$.

Let ϕ be a mapping defined on S' as follows:

$$\phi(x) = y \text{ s.t. } RANK_1(x, S') = RANK_2(y, S').$$

Note that ϕ is a symmetric permutation operator on S'.

Now construct a new bargaining problem with preferences defined as follows $(S'', \succeq'')$ where $S' = S''$ and $\succeq''$ is induced from the lower contour sets defined by:

$$LCS_i(x) \equiv co(\{\ell \not\succeq'_i x\} \cup \phi(\{\phi(\ell) \not\succeq'_j \phi(x)\})).$$

where *co* denotes the convex hull. By taking the convex hull of the "not better than sets", the preferred sets to any nondegenerate allocation are weakly contracted for both agents.

By construction if $\ell \succeq''_i x$ then both $\ell \succeq''_i x$ and $\phi(\ell) \succeq''_j \phi(x)$, and since by definition ϕ is one to one, $\phi(\phi(\ell)) = \ell$ and so we have that $\phi(\ell) \succeq''_j \phi(x)$. Thus $(S'', \succeq'')$ is a symmetric problem with respect to ϕ. [The proof that $(S'', \succeq'')$ is in the domain Σ_A is available on request.]

We next prove that if a solution F satisfies SYM and WCO, then $F(S'', \succeq'') = OE(S'', \succeq'')$.

1. Consider the case in which the OE solution is a degenerate lottery, that is $OE(S, \succeq) = x$. Observe that in this case, by construction, we also have that $OE(S'', \succeq'') = x$.

Now consider the problem $(S'', \succeq''')$. By construction, $DPO(S'', \succeq'') = DPO(S, \succeq) = S''$. Thus, since x is the OE solution of S'' it is by definition the "middle point" of S''. It follows that x is a fixed point of the symmetry mapping ϕ on S''. Since by Lemma 2, the OE solution satisfies WCO we also have that $x \in WCO(S', \succeq')$. If x is the **unique** WCO fixed point, then by SYM and WCO the $x = F(S'', \succeq'')$.

Suppose instead that x is **not** the unique WCO fixed point in S'' under ϕ. In the following we demonstrate that by perturbing preferences and applying the IRA axiom such lotteries cannot be the solution. Since x is the unique fixed point in S'' (that is, degenerate lottery which is a fixed point), any other symmetric point must be a nondegenerate lottery ℓ'. By the construction of the symmetry mapping ϕ, the support of such of a lottery, $supp(\ell')$ must contain points c, d such that $c \succeq_1 x \succeq_1 d$. Suppose it happens that $\ell' = (\frac{1}{2}, \frac{1}{2}; c, d)$ and without loss of generality $\ell' \succeq_1 x$. But then by the Archimedean axiom x is the certainty equivalent of some lottery $\ell_1 = (\mu, 1-\mu; c, d)$ (with $\mu \leq \frac{1}{2}$), and so $\ell' = (\frac{1}{2}, \frac{1}{2}; c, d)$ can be written as a compound lottery over c and ℓ_1. Thus, by the ORA axiom, using the construction used in the proof of Lemma 2, as c and d are Pareto optimal, there must exist another lottery $\ell^* = (\lambda, 1-\lambda; c, x)$ that Pareto dominates or is Pareto equivalent to $(\frac{1}{2}, \frac{1}{2}; c, d)$. Now define the problem $(\tilde{S}, \tilde{\succeq})$ where $S'' = \tilde{S}$ and $\tilde{\succeq}$ has the same ordinal structure as $\succeq''$ but is such that ℓ^* strictly Pareto dominates $(\frac{1}{2}, \frac{1}{2}; c, d)$. That is we have made certain lotteries "redundant" in the sense of IRA axiom. Then, by WCO, as ℓ is dominated by a Pareto optimal degenerate allocation, $\ell \neq F(S'', \succeq'')$.

The same argument holds regardless of how many points are in the candidate symmetric, WCO lottery (ℓ' in the paragraph above) and so we can conclude that no alternative symmetric lottery could be the solution to $(S\ ,\succeq\ )$ Thus, for the problem $(S\ ,\succeq\ )$ the point x is the unique fixed point that is also in $WCO(S\ ,\succeq\ )$, and thus $x = F(S\ ,\succeq\ )$. Now, since the hypothesis of IRA applies to the pairs $(S'', \succeq'')$, and $(S\ ,\succeq\ )$ we have that $x = F(S'', \succeq'')$. However, by the construction of $(S', \succeq')$ as the hypothesis of IRA also applies to the pairs $(S, \succeq), (S', \succeq')$ it follows that by repeated application of IRA that $F(S, \succeq) = F(S'\ \succeq') = F(S'', \succeq'') = x = OE(S, \succeq)$.

2. Now consider the case in which the OE solution is a non-degenerate lottery, that is $OE(S, \succeq) = \ell$. Essentially, the same argument given above holds:

Let (a, b) be the support of the points in lottery $\ell = OE(S', \succeq')$. Let (c,d) be two points in the support of an alternative symmetric, WCO lottery, ℓ', where without loss of generality $c \succeq_1 a \succeq_1 b \succeq_1 d$.

Let ℓ^a be the lottery over (c, d) that is exactly as good as point a to agent 1.

Let ℓ^b be the lottery over (c, d) that is exactly as good as point b to agent 1.

We then apply ORA to the 50/50 lottery over ℓ^a and ℓ^b in the same way we do above, and the argument proceeds as before.

We conclude that a solution on Σ_A satisfies SYM, WCO and IRA if and only if $F = OE$.

■

Finally, if we are willing to further restrict the domain of preferences to those satisfying ordinal risk neutrality, we can use Corollary 1 to show that the characterization can be strengthened to include full WPO instead of WCO.

Theorem 3. *A solution on Σ_N satisfies SYM, WPO and IRA if and if $F = OE$.*

Proof/

The proof is almost identical to that for Theorem 2. To begin, by Lemmas 4, and 3, the OE solution satisfies SYM, and IRA. Also, on Σ_N, by Corollary 1 the OE solution satisfies WPO.

To show the converse, note that any solution that satisfies WCO also satisfies WPO. Thus we can apply the previous theorem directly.

■

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