Implementing the Nash Extension Bargaining Solution†
for Non-convex Problems

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† **Published as**: John P. Conley and Simon Wilkie "Implementing the Nash Extension Bargaining Solution for Non-Convex Problems", *Economic Design*, Vol. 1, No. 2, 1995, pp. 205-216.

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Abstract

Conley and Wilkie (1993) introduced and axiomatized the Nash extension bargaining solution, defined on a domain of comprehensive but not necessarily convex problems. In this paper we present a non-cooperative game that implements the Nash extension solution in subgame perfect equilibria in the limit as the discount rate applied between rounds of play vanishes.

Journal of Economic Literature Classification Numbers: C71, C72.

Key words: Implementation, Cooperative Games, Nash Bargaining Solution, Bargaining Theory, Non-convex Economies.

1. Introduction

An *n-person bargaining problem* consists of a pair (S, d) where S is a non-empty subset of $\Re^n$, and $d \in S$. We interpret S as the set of feasible utility allocations that are attainable through joint action on the part of all n agents. If the agents fail to reach agreement on the division of the surplus, then the problem is settled at the point d, which is called the *disagreement point.* A *bargaining solution* F, defined on a class of problems Σ^n, is a map that associates with each problem $(S, d) \in \Sigma^n$ a unique point in S.

In the cooperative, or *axiomatic* approach to bargaining, we abstract from the bargaining procedure itself, and specify a list of properties that a bargaining solution should satisfy. If there is a unique solution that satisfies a given list, then the solution is said to be *characterized* by these axioms. Nash (1950), initiated this approach by proposing four axioms and proving that they characterize what is now known as the Nash bargaining solution. Nash (1953), then turned to the question of how this solution might be obtained in the real world given the constraint that agents are concerned only with maximizing their own welfare. His answer was to design a noncooperative game in which the only equilibrium outcome is exactly the allocation suggested by the Nash solution. Unfortunately, Nash's procedure requires the designer to have knowledge of the pair (S, d), and by implication, the agents' utility functions. This is an unrealistically strong assumption. In general, we would prefer to specify rules for a noncooperative game, such that for any profile of the agent's utilities unknown to the designer, the equilibrium payoff vector is unique, and coincides with the allocation chosen by a particular solution. If such a game exists, we say it *implements* the solution concept.

In a seminal paper, Rubinstein (1982) examined the subgame perfect equilibria of an alternating offers bargaining procedure. Subsequently Binmore, *et al* (1986) proved that as the time between offers became arbitrarily small, the equilibrium payoffs of this game converge to the Nash solution of the associated bargaining problem. Thus, the Rubinstein bargaining procedure implements the Nash bargaining solution. Similarly, Moulin (1984), proposed a simple bargaining procedure and proved that the unique subgame perfect equilibrium payoff was exactly the outcome chosen by the Kalai-Smorodinsky (1975) bargaining solution.

In the above literature, both the axiomatic and implementation papers restrict attention to problems with convex feasible sets. This restriction is usually defended by assuming von Neumann-Morgenstern preferences and allowing the use of lotteries. However, several recent papers have considered how to settle nonconvex bargaining problems without resorting to lotteries. In particular, Kaneko (1980), Herrero (1989), and Conley and Wilkie (1993) study extensions to the Nash solution defined on the class of comprehensive, but not necessarily convex bargaining problems. Other solutions are studied in Anant et al (1990) and Conley and Wilkie (1991). In Conley and Wilkie (1993), we argue that even if agents have von Neumann-Morgenstern preferences, allowing the problem to be settled at an allocation attainable only by use of a lottery may be inappropriate or even impossible in many situations in which bargaining solutions are traditionally applied. Simply put, this is because when a problem is settled at a lottery, the axioms that characterize the solution are only satisfied in expectation, and not, in general, by the final utility allocations which agents receive after the lottery is held. If we wish the axioms to apply to the ex-post allocations, then the solution should be required to select a point in the feasible set that does not require the use of lotteries.

Each of the papers above provides an axiomatic characterization for the solution concept it considers. Herrero (1989), is also able to show that the stationary subgame perfect equilibrium outcomes of the Rubinstein bargaining procedure converge to the set of allocations chosen by her proposed solution. Thus, she is able to implement her bargaining solution. In Conley and Wilkie (1993), we introduced and axiomatized a new bargaining solution, the Nash extension solution, defined on the space of bargaining problems with comprehensive, but not necessarily convex feasible sets. Unlike the proposals of Kaneko (1980), and Herrero (1989), the Nash extension solution is single-valued and continuous. In this paper we provide a noncooperative foundation for the

Nash extension solution by constructing a game that implements the Nash extension solution on a nonconvex domain in subgame perfect equilibria.

The game we construct requires agents to bid for first mover advantage in the manner of Moulin's (1984) auction game (which implements the Kalai–Smorodinsky solution). Subsequently, agents can make offers in the manner of the Rubinstein–Binmore bargaining game (which implements the Nash solution. Our game bears a resemblance to the game presented in Howard (1992) which implements the modified Nash solution discussed in Luce and Raiffa (1957). This modified solution differs from the standard Nash solution in that the role of the disagreement point is played by the expected random dictatorship payoff. Our game differs from Howard's in several important respects. In particular: (a) we require players to announce allocations attainable *without* the use of lotteries in the first round of play, (b) we use an exogenous prespecified disagreement point instead of an endogenously determined random dictator point to define our solution concept, and (c) the third stage of our game is Rubinstein's alternating offers game rather than a "random dictatorship game." It is interesting to note, however, that if we replace the first stage of Howard's game with the first stage of our game then this new game implements a "modified Nash extension" solution.

A brief outline of the paper is as follows: we present some technical definitions in section 2, the solution is introduced in section 3, and the game implementing the solution is presented in section 4.

2. Definitions

We start with some definitions. As the game provided in section 4 implements the Nash extension for the two agent case, we restrict attention to $\Re^2$ below. Given a point $d \in \Re^2$, and a set $S \subset \Re^2$, we say S is *d-comprehensive* if $d \leq x \leq y$ and $y \in S$ imply $x \in S$.[1] The *comprehensive hull of a set* $S \subset \Re^2$, *with respect to a point* $d \in \Re^2$ is the smallest d-comprehensive set containing S:

$$comp(S; d) \equiv \{x \in \Re^2 \mid x \in S \text{ or } \exists\, y \in S \text{ such that } d \leq x \leq y\}. \tag{1}$$

The *convex hull of a set* $S \subset \Re^2$ is the smallest convex set containing the set S:

$$con(s) \equiv \left\{x \in \Re^2 \mid x = \lambda y + (1-\lambda)y' \text{ where } \lambda \in [0,1] \text{ and } y, y' \in S\right\}. \tag{2}$$

Define the *weak Pareto frontier* of S as:

$$WP(S) \equiv \{x \in S \mid y \gg x \text{ implies } y \notin S\}. \tag{3}$$

Define the *strong Pareto frontier* of S as:

$$P(S) \equiv \{x \in S \mid y \geq x \text{ implies } y \notin S\}. \tag{4}$$

The domain of bargaining problems considered in this paper is Σ_c. This is defined as the class of pairs (S, d) where $S \subset \Re^2$ and $d \in \Re^2$ such that:

A1) S is compact.

A2) S is d-comprehensive.

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[1] The vector inequalities are represented by $\geq, >, and \gg$.

A3) There exist $x \in S$ and $x \gg d$.

This differs from the usual domain, which we denote Σ_{con}, in that we do not assume that the set of feasible utility allocations is convex. A *bargaining solution*, F, is a function from a class of problems Σ to $\Re^2$ such that for each $(S,d) \in \Sigma$, $F(S,d) \in S$. The *ideal point* of a problem (S,d) is defined as:

$$a(S,d) \equiv (\max_{\substack{x\in S\\ x\geq d}} x^1, \max_{\substack{x\in S\\ x\geq d}} x^2). \tag{5}$$

The *ethical point with respect to a solution F* of a problem (S,d), is defined as:

$$e^F(S,d) \equiv F(con(S),d). \tag{6}$$

This is simply the point recommended by the solution F to the convex hull of the problem (S,d). See Conley and Wilkie (1993) for further motivation of the ethical point.

3. The Nash Extension Solution

In his 1950 paper, Nash (1950) considered the domain Σ_{con} of convex problems. He proposed the following solution:

$$N(S,d) \equiv \left\{ \operatorname*{argmax}_{\substack{x\in S\\ x\geq d}} (x_1 - d_1)(x_1 - d_2) \right\}, \tag{7}$$

and proved that it is the unique solution that satisfies Nash's four axioms. In a subsequent paper, Nash (1953), he introduced the "Nash demand game" and was able to show that the Nash equilibria of the demand game approximated the Nash bargaining solution of the associated bargaining problem.

Nash's demand game suffers from a serious drawback from the standpoint of implementation theory. Nash requires that the designer of the game know the set, S, and hence the agent's utility functions. Thus, although the Nash demand game provides a noncooperative foundation for the Nash bargaining solution, it does not solve the implementation problem. An alternative to Nash's game is the alternating offers game introduced in Rubinstein (1982). Subsequently, Binmore *et al* (1988) proved that the unique equilibrium outcome of this game converges to the Nash solution payoffs of the associated bargaining game. The Rubinstein game does not require the designer of the game to know the agents' utilities, and so, (in the limit) it implements the Nash bargaining solution.

Extensions of the Nash solution to the domain of nonconvex problems include Kaneko (1980), and Herrero (1989). Kaneko offers a characterization of the direct generalization of the Nash solution, the set of Nash product maximizers, on the domain of "regular" bargaining problems. However, Kaneko's solution is not single-valued and is only upper-hemicontinuous. Herrero (1989) also provides a set of axioms, and defines a set-valued generalization of the Nash solution on the strictly comprehensive two person domain. Her solution selects the set of "local" maximizers of the Nash product, which is a superset of the outcomes recommended by Kaneko's solution. Herrero provides an axiomatic characterization of his solution, and proves that this solution is (in the limit) the set of subgame perfect equilibria of the Rubinstein alternating offers game for bargaining problems with nonconvex feasible sets. Thus, she is able to implement her solution in (limit) subgame perfect equilibria.

Like Herrero's and Kaneko's solutions, the Nash extension solution, introduced in Conley and Wilkie (1993), coincides with the Nash solution when the feasible set is convex. Unlike the other generalizations to nonconvex bargaining problems, however, the Nash extension solution is single-valued and continuous.

We construct the Nash extension as follows: First define the mapping $L : \Sigma_c \to \Re^2$ as:

$$L(S,d) \equiv con\left(e^N(S,d), d\right). \tag{8}$$

$L(S,d)$ is the line segment connecting the disagreement point d, to the Nash solution of the problem composed of the convex hull of S (the ethical point), and d. Now we define the solution NE:

$$NE(S,d) \equiv \{\max x \mid x \in L(S,d) \cap S\}, \tag{9}$$

Insert figure 1 about here

where max indicates the maximal element with respect to the partial order on $\Re^2$. The construction of NE is illustrated in figure 1. The point $NE(S,d)$ is the intersection of the weak Pareto frontier of S and the line segment connecting the disagreement point and ethical point under the Nash solution. Obviously, NE coincides with N on the domain of convex problems. A brief motivation for this solution concept is the following. We would like to settle the bargaining problem at the Nash point, however, if the underlying feasible set is nonconvex, this requires the use of lotteries. We reject this outcome because the allocation satisfies the axioms that characterize the solution only in expectation, and not after the lottery is held. Our compromise is to choose the largest feasible allocation on the line between the Nash point and the disagreement point. In essence, we avoid the use of a lottery by compromising away from the ethically most desirable point (as defined by the axioms) in a way that shares the losses over the agents. In Conley and Wilkie (1993), we provide a more extensive motivation, show that the NE solution is nonempty, single-valued and continuous, and provide an axiomatic characterization.

4. Implementing the Nash Extension Solution

In this section, we present a noncooperative game which implements the Nash extension solution. We consider an environment with two agents and a (neutral) mediator. The agents act purely out of self-interest, and can communicate only through the mediator. The agents' utility functions, however, are not known to the mediator. This is also the problem studied by Moulin (1984), and Howard (1992).

More formally, we suppose that the two agents, 1 and 2, have available a set of real economic alternatives, A, with a distinguished element, $a^d \in A$, as the disagreement alternative. The agents have utility functions u_1 and u_2 defined over A which yields the bargaining problem (S, d), where $S \subset \Re^2_+$ is the image A under the map (u_1, u_2) and d is the image of a^d. We assume that the pair $(S, d) \in \Sigma_c$. ¿From now on utilities are normalized such that $d = (0, 0)$. We assume that S is strictly comprehensive and so there exists a continuous, strictly decreasing function $f : \Re_+ \to \Re_+$, where $f(x) = max\{y \in \Re_+ | (x, y) \in S\}$. We also assume that preferences satisfy the von-Neumann Morgenstern axioms and so we may extend u_i to $\mathcal{M}$, the space of lotteries defined over A. We denote by $M \subset \Re^2_+$ the image of $\mathcal{M}$ under the maps u_1, u_2. The pair (M, d) is also a bargaining problem; indeed, it is one with a convex feasible set.

Recall that Maskin (1977) identified the following necessary and (almost) sufficient condition for a choice function to be implementable in Nash equilibria. On our domain Maskin's condition is defined as:

Maskin Monotonicity: Let $u = (u_1, u_2)$ and $u' = (u'_1, u'_2)$, be two arbitrary pairs of utility functions from a given class of utility functions, and define $(S, d) = (u(A), u(a_d))$, and $(S', d') = (u'(A), u'(a_d))$. A solution F, is Maskin monotonic if for all such pairs of utility functions, $u(a') = F(S, d)$ and for all i, $\{a \in A \mid u_i(a) \geq u_i(a')\} \subset \{a \in A \mid u'_i(a) \geq u'_i(a')\}$, implies $u'(a') = F(S', d')$.

Thus, the first question which arises in implementing a solution is, is the solution Maskin-monotonic? The following example shows that the Nash-extension solution is not Maskin monotonic.

Let $A = \{(x, y) \in \Re^2 \mid x + y \leq 1, x \geq 0, y \geq 0\}$, and $a_d = (0, 0)$. Also let $u = (x, y)$ and $u' = (x, y^2)$. It is easy to verify that the pair $u.u'$ satisfy the hypothesis of Maskin's condition, and that the allocation $a' = (\frac{1}{2}, \frac{1}{2})$ is such that $u(a') = NE(S, d)$. However $u'(a') \neq NE(S', d')$. Thus the solution is not Maskin monotonic, and hence is not implementable in Nash equilibria.

Since we cannot implement in Nash equilibrium, and in particular, cannot use Maskin's game, we must provide a new game and use a different equilibrium concept. We will use subgame perfect equilibria as our implementation concept for the mechanism defined below. General conditions for implementation have been introduced by Moore and Repulo (1988), and Abreu and Sen (1990). Formally, the result we obtain is similar to those of Binmore *at al* (1988) implementing the Nash solution, and Herrero (1989) implementing her set-valued extension of the Nash solution. That is, the limit of the (unique) subgame perfect equilibrium payoffs, as the discount parameter goes to one, is the utility allocation prescribed by the Nash extension solution.

Recall the definition of the Rubinstein alternating offer game. Time is discrete. In each time period, t, one agent gets to propose an allocation in S to the other, who can either accept or refuse. If the second agent refuses, the feasible set is discounted at rate δ, and in the next period he makes a proposal. If this proposal is refused, the feasible set is again discounted, and agent one makes a proposal. The game so proceeds with the roles alternating until a proposal is accepted. If no proposal is ever accepted, the agents obtain the disagreement point.

Let $\delta \in (0, 1)$ be given, and define the game $\Gamma(\delta)$ as follows. The game has three stages. In the first, or auction stage, each agent announces a number in the unit interval. The agent who announces the higher number then proposes an allocation. The other agent can accept or reject the offer. If the proposal is accepted it is the end of the game. If it is rejected the game enters the second, or lottery stage, where the neutral mediator with probability equal to the lower bid continues the game, and otherwise enforces the disagreement outcome. In the third, or bargaining stage, the agents play Rubinstein's alternating offers game, where the second agent makes the first proposal. Again the mediator enforces outcomes, including any required randomizations. Formally, we define the game $\Gamma(\delta)$ by:

Stage 1:

Each agent chooses a number from the set $P_i = [0, 1]$. If $p_1 > p_2$ then 1 proposes an allocation, $a_1 \in A$. Agent 2 announces *"Yes"* or *"No"*.

If agent 2 announces *"Yes"*, the game ends with payoffs $u_1(a_1), u_2(a_1)$. If agent 2 announces *"No"*, then the game proceeds to Stage 2.

If $p_1 < p_2$, then the roles are reversed, agent 2 gets to propose an allocation.

If $p_1 = p_2$, then a fair coin is tossed to decide who proposes an allocation.

Stage 2:

If $p_1 > p_2$, then with probability $1 - p_2$ the bargaining process ends and the outcome is $u_1 = u_2 = 0$. With probability p_2, the game proceeds to Stage 3.

If $p_1 \leq p_2$, then with probability $1 - p_1$ the bargaining process ends and the outcome is $u_1 = u_2 = 0$. With probability p_1, the game proceeds to Stage 3.

Stage 3:

If $p_1 > p_2$, then agent 2 is chosen as the leader in Rubinstein's alternating offer game with discount factor δ applied between offers. Agent 2 proposes an element in $\mathcal{M}$. Agent 1 can accept or reject the offer. If he accepts, then the offer is implemented with the mediator enforcing the outcome, including any required randomization. If agent 1 rejects 2's offer, then from the next time period, play proceeds with 1 making a proposal from $\mathcal{M}$.

If $p_1 < p_2$, then the agents' roles above are reversed.

If $p_1 = p_2$, then a fair coin is tossed to decide who makes the first offer.

The astute reader will have noticed that the agents' choice sets change from Stage 1 to Stage 3. In Stage 1 of the game, players are constrained to announce allocations attainable without the use of lotteries. If the game reaches the third stage on the other hand, the mediator may enforce an allocation attainable only through randomizations. This construction may seem artificial and inconsistent with our stated purpose of implementing a solution that does not require the use of lotteries. The reason that this is acceptable here is that in equilibrium lotteries will never be used. From the point of view of implementation theory, the designer of the game is concerned only with the equilibrium outcomes. The fact that the game that the game form requires the use of randomizations off the equilibrium path is irrelevant. The feature that off the equilibrium path a game may give outcomes that are highly inconsistent with the objectives of the mechanism designer is common in implementation literature. For example, the games used in Maskin (1977), and Moore and Repullo (1988) can be used to implement efficient and equitable social choice rules even though out of equilibrium they yield outcomes that will never be efficient or equitable. For a critique these features in implementation theory see Chakravorti, Corchon and Wilkie (1994).

In the following we will always mean "subgame perfect equilibrium" when we say "equilibrium." We begin with some well known facts.

Fact 1. In the Rubinstein sub-game, when 1 has the first offer and 2 replies, for each value of $\delta < 1$, there exists a unique equilibrium payoff pair, $(x_1^o(\delta), x_2^r(\delta))$, and when 2 has the first offer and 1 replies, there exists a unique equilibrium payoff pair, $(x_1^r(\delta), x_2^o(\delta))$.

Fact 2. $Lim_{\delta\to 1}(x_1^o(\delta), x_2^r(\delta)) = Lim_{\delta\to 1}(x_1^r(\delta), x_2^o(\delta)) = N(M, d)$.

For a proof of Facts 1 and 2 see van Damme (1987), Theorems 7.6.5 and 7.6.7.

We now introduce some notation.

Let $x(\delta) = (x_1(\delta), x_2(\delta)) = (x_1^o(\delta), x_2^o(\delta))$. Let $\hat{p}(\delta)$ be defined as the maximal value of $p \in \Re_+$ such that $p \cdot x(\delta) \in S$. Because both $(x_1^o(\delta), x_2^r(\delta))$ and $(x_1^r(\delta), x_2^o(\delta))$ are Pareto-optimal, and, as $\delta \leq 1$, then $(x_1^o(\delta), x_2^r(\delta)) \neq (x_1^r(\delta), x_2^o(\delta))$, we have that $x(\delta) \notin S$. Thus for all (δ), $\hat{p}(\delta) \leq 1$. Let $(e_1(\delta), e_2(\delta)) = (\hat{p}(\delta) \cdot x_1(\delta), \hat{p}(\delta) \cdot x_2(\delta))$.

Lemma 1. *For all $\delta < 1$ the game $\Gamma(\delta)$ has a unique subgame perfect equilibrium payoff, $(u_1^*, u_2^*) = (e_1(\delta), e_2(\delta))$.*

Proof/

First we describe the equilibrium payoffs for various configurations of bids by agents in the first stage.[2]

If $p_1 > p_2$ and the game reaches stage three, then from Fact 1 the unique subgame perfect equilibrium continuation payoffs are (x_1^r, x_2^o). Since $p_1 > p_2$, agent 1 makes an offer to agent 2 in stage 1. By subgame perfection, agent 2 will accept an offer $a \in A$ for which $u_2(a) \geq p_2 x_2^o$. Thus the best offer from agent 1's perspective that is sure to be accepted by agent 2 is $a \in A$ such that $u_1(a) = f^{-1}(p_2 x_2^o)$, and $u_2(a) = p_2 x_2^o$.

If $p_1 < p_2$ then by symmetric argument, agent 2 makes the best offer that he knows that agent 1 will accept. In this case the offer is $a \in A$ such that $u_1(a) = p_1 x_1^o$ and $u_2(a) = f(p_1 x_1^o)$.

Finally, if $p_1 = p_2 \equiv p$ then a fair coin is tossed to see who gets to make the first offer and payoffs are the average of the payoffs described above: $(\frac{1}{2}(f^{-1}(px_2^o) + px_1^o), \frac{1}{2}(f(px_1^o) + px_2^o))$.

We now show that e is indeed an equilibrium payoff.

Consider the following strategies. Agent 1 announces $p_1 = \hat{p}$. Then if selected to offer he proposes an allocation a' such that $u_2(a') = p_2 x_2^o$ and $u_1(a') = f^{-1}(p_2 x_2^o)$. If the offer is rejected and the game does not end in stage 2, then in stage 3 she rejects any proposed lottery l, if $u_1(l) < x_1^r$, and accepts it otherwise. Whenever it is his turn to propose he suggests a lottery l' that yields $(u_1(l'), u_2(l')) = (x_1^0, x_2^r)$. Agent 2 announces $p_2 = \hat{p}$. Then if selected to offer he proposes an allocation a' such that $u_1(a') = p_1 x_1^o$ and $u_2(a') = f(p_1 x_2^o)$. If the offer is rejected and the game does not end in stage 2, then in stage 3 she rejects any proposed lottery l, if $u_2(l) < x_2^r$, and accepts it otherwise. Whenever it is his turn to propose he suggests a lottery l' that yields $(u_1(l'), u_2(l')) = (x_1^r, x_2^o)$.

It is easy to verify that the above strategies form a subgame perfect equilibrium. Applying Fact 1 above, given p_1 and p_2, there is a unique equilibrium continuation payoff in stage 3. Backward induction tells us the minimum level of utility an agent will accept, given his bid p_i. It is then easy to show that there is no gain to changing

[2] As δ is fixed in the following, we suppress its appearance as an argument.

p_i, given $p_j = \hat{p}$. Thus the strategies are indeed subgame perfect, and yield the payoffs $(\hat{p}x_1^o, f(\hat{p}x_1^o))$, but by the definition of $\hat{p}$, this is exactly (e_1, e_2).

We now show that this is indeed the only equilibrium outcome. Again from Fact 1, once stage 3 is reached, there is a unique subgame perfect continuation payoff. It thus remains to show that $p_1 = p_2 = \hat{p}$ is the only possibility in stage 1.

Suppose that $\hat{p} > p_1$. The hypothesis implies that $(p_1 x) \in int(S)$. Therefore there exists $a' \in A$ such that, $u_1(a') = p_1 x_1^o$ and $u_2(a') = f(p_1 x_1^o) > p_1 x_2^o$. However if agent 2 announces $p_2 \leq p_1$, and 1 is chosen to offer, by subgame perfection, his payoff is at most $p_2 x_2^o \leq p_1 x_2^o$. Thus agent 2 will always announce a higher p than agent 1 to get this first mover advantage. The symmetric argument holds for agent 1 if $\hat{p} > p_2$. We conclude that it is never part of an equilibrium strategy to announce a p_i smaller than $\hat{p}$.

Now suppose that $\hat{p} < p_1$. The hypothesis implies that $(p_1 x) \notin S$. If $p_2 > p_1$, then 2 proposes an allocation to 1. By subgame perfection, 1 will reject any offer a with $u_1(a) < p_1 x_1^o$. By construction, as $p_1 > \hat{p}$, $f(p_1 x_1^o) < p_1 x_2^o$. Furthermore $x_2^r < x_2^o$. Thus the highest payoff that 2 can attain if $p_2 \geq p_1$ is the maximum of $f(p_1 x_1^o)$ and $p_1 x_2^r$. However, by hypothesis there exists a $p_2' < p_1$ which ensures 2 a payoff of a least $p_2' \delta x_2^o(\delta) > Max[f(p_1 x_1^o), p_1 x_2^r]$. Thus if $p_1 > \hat{p}$, then agent 2's best response must be $p_2 < p_1$. The symmetric argument holds for agent 1 if $p_2 > \hat{p}$. Thus in any equilibrium it must be that $p_1 = p_2 = \hat{p}$. Then the unique subgame perfect equilibrium payoffs are: $(\hat{p} \cdot x_1(\delta), \hat{p} \cdot x_2(\delta)) \equiv (e_1(\delta), e_2(\delta))$.

■

The main result of the paper is:

Theorem 2. *The limit as $\delta \to 1$ of the subgame perfect equilibrium payoffs of $\Gamma(\delta)$ is exactly the Nash-extension solution to the bargaining problem (S, d).*

Proof/

From Lemma 1 we have that for all δ, the subgame perfect equilibrium payoffs of $\Gamma(\delta)$ are $(e_1(\delta), e_2(\delta))$. From Fact 2 above, we know that

$$Lim_{\delta\to 1}(x_1^o(\delta), x_2^r(\delta)) = Lim_{\delta\to 1}(x_1^o(\delta), x_2^r(\delta)) = N(M, d). \tag{10}$$

Thus we have that $x(\delta) \to N(M, d)$. Furthermore $\delta S \to S$, and therefore $\delta M \to M$. Thus from the definitions of $\hat{p}(\delta)$ and $NE(S, d)$, $\hat{p}(\delta) \to \mu \leq 1$, where $NE(S, d) = \mu N(M, d)$. Therefore, by the continuity of multiplication on $\Re^2$, $e(\delta) \to \mu N(M, d) = NE(S, d)$.

■

5. Conclusion

In Conley and Wilkie (1993) we introduced the Nash extension solution defined on the space of comprehensive, but possibly nonconvex sets. There, we argued that the axioms used to characterize the NE solution suggest that it is a hybrid of the Nash bargaining solution, and the Kalai-Smorodinsky bargaining solution. In this paper we introduce a game that implements the Nash extension bargaining solution in (limit) subgame perfect equilibria. The game has three stages. The first two are essentially the same as the stages in Moulin's bargaining game, which implements the Kalai-Smorodinsky solution. The third stage is Rubinstein's alternating offering game, which implements the Nash bargaining solution. Thus, the game form used to implement the solution further underscores how the Nash extension solution can be seen as a hybrid of the Nash and the Kalai-Smorodinsky bargaining solutions.

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