An Extension of the Nash Bargaining Solution†
to Nonconvex Problems

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Abstract

We investigate the domain of comprehensive but not necessarily convex bargaining problems. Without convexity the Nash solution is not well defined. We propose a new solution, the *Nash Extension,* that coincides with the Nash solution when S is convex. We characterize it by Weak Pareto Optimality, Symmetry, Scale Invariance, Continuity, and a new axiom, *Ethical Monotonicity.*

1. Introduction

An *n-person bargaining problem* consists of a pair (S, d) where S is a nonempty subset of $\Re^n$, and $d \in S$. The set S is interpreted as the set of utility allocations that are attainable through joint action on the part of all n agents. If the agents fail to reach an agreement, then the problem is settled at the point d, which is called the *disagreement point*. A *bargaining solution*, F, defined on a class of problems Σ, is a map that associates with each problem $(S, d) \in \Sigma$ a unique point in S. In the axiomatic approach to bargaining we start by specifying a list of properties that we would like a solution to have. If it can be shown that there is only one solution that satisfies a given list of axioms, then the solution is said to be *characterized* by this list.

In this paper we provide an extension of the Nash solution to the domain of comprehensive (assured, for example, by free disposal), but not necessarily convex bargaining problems. In section two, we motivate the study of nonconvex problems. In section three we define the axioms and mathematical objects used in subsequent sections. In section four we propose a new solution called the *Nash Extension* and provide it with an axiomatic characterization. We discuss why the Nash solution is not well defined on this domain, and argue that our extension has several important properties which alternative generalizations in the literature fail to satisfy. Elsewhere, Conley and Wilkie (1993), we present a noncooperative game that implements the new solution. There our result is akin to those of Binmore *et al* (1986), and Herrero (1989). We show that in the limit, the unique subgame perfect equilibrium payoff is exactly the utility allocation prescribed by our proposed solution to the cooperative game. Section five concludes

2. Nonconvex problems

The bargaining literature traditionally has focused on problems with convex feasible sets. However, nonconvex problems arise from many real economic situations. For example, the projection on utility space of a finite set of goods allocations will be a finite number of points. Thus, the feasible set may be nonconvex if it is derived from a world with a finite number of large public projects, or with indivisibilities in the private goods. Externalities in production, and in consumption, may also lead to a nonconvex feasible set (see Starrett, 1972). It is usually argued that if agents have von Neumann-Morgenstern utility functions, then these primitive feasible sets can be "convexified" by running lotteries over the initial set of allocations. Thus, it is claimed that if one accepts the von Neumann-Morgenstern axioms on preferences, then by using lotteries, the convex hull of the *primitive* feasible set becomes the *true* feasible set.

This argument has increasingly come under question in the recent literature. In particular Binmore (1987) argues that more attention should be paid to the structure of the primitive set, and Rubinstein, Safra and Thomson (1992) relax the assumption that agents have von Neumann-Morgenstern utilities. Several papers in utility theory, surveyed in Machina (1987), have relaxed the independence axiom. Without this axiom utility is no longer linear in the probabilities, thus merely invoking lotteries does not ensure that the feasible set is convex. Most relevant for our purpose is Kreps-Porteus (1979) and Machina (1984), who show that even if the underlying preferences satisfy the von Neumann-Morgenstern axioms, when there is a temporal nature to decision making the induced utility functions will be quasiconvex, and so the appeal to lotteries must fail if there is a temporal element in the bargaining.

Several recent papers have considered how to settle nonconvex bargaining problems. In particular, Anant et al (1990) and Conley and Wilkie (1991) study the solution proposed by Kalai and Smorodinsky (1985), Conley and Wilkie (1991) study the egalitarian solution, and Kaneko (1980) and Herrero (1989) study the Nash solution. All of these papers do not explicitly reject the hypothesis of von Neumann-Morgenstern preferences, and it is not immediately obvious how the refusal to use lotteries to con-

vexify feasible sets be reconciled with this assumption on preferences. It appears not to respect agents' preferences, and opens the possibility that bargaining problems may not be settled at Pareto efficient allocations.

We offer the following explanation for this seeming contradiction. In many applications of bargaining theory, allowing the problem to be settled at a point attainable only by the use of a lottery is inappropriate or even impossible. Simply put, this is because when a problem is settled at a lottery, the axioms that characterize the solution are satisfied only in expectation, and not (in general) by the final allocation the agents receive after the lottery is resolved. In other words, even if we accept the von Neumann-Morgenstern assumptions on preferences, and therefore believe that the convex hull is a payoff relevant object, we still must decide whether the axioms that characterize the solution need to be satisfied by the ex-post allocations or if it is sufficient that they be satisfied by the ex-ante allocations.

This question is important if bargaining theory is viewed as a method of prescribing "fair" or "ethical" settlements to social distribution problems. Here ex-ante/ex-post dichotomy may be viewed as a choice between fairness of opportunity and fairness of result.[1] Alternatively, following Zuethan and Harsanyi, bargaining theory may also be used to help understand certain noncooperative situations. This type of application is predicated on the belief that if agents are persuaded that a particular division is "fair", then they will voluntarily agree to coordinate their actions and accept the allocation as a settlement to their problem. Here, the decision to allow settlements that are only ex-ante fair depends on whether or not the agents can sign binding contracts to abide by the outcomes of lotteries or to make ex-post transfers of wealth. Thus, even when agents have von Neumann-Morgenstern preferences, the suggested allocation will not generally satisfy the axioms after the lottery is held, and so without binding contracts such a settlement will break down ex-post.

3. Definitions and Axioms

We start with some definitions and formal statements of the axioms used. Given a point $d \in \Re^n$, and a set $S \subset \Re^n$, we say S is *d-comprehensive* if $d \leq x \leq y$ and $y \in S$ implies $x \in S$.[2] The *comprehensive hull of a set $S \subset \Re^n$, with respect to a point $d \in \Re^n$* is the smallest d-comprehensive set containing S:

$$comp(S; d) \equiv \{x \in \Re^n \mid x \in S \text{ or } \exists \, y \in S \text{ such that } d \leq x \leq y\}.$$

The *convex hull of a set $S \subset \Re^n$* is the smallest convex set containing the set S:

$$con(S) \equiv \left\{ x \in \Re^n \mid x = \sum_{i=1}^{m} \lambda_i y_i \text{ where } \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geq 0 \, \forall \, i, \text{ and } y_i \in S \, \forall \, i \right\}.$$

Let $\mathcal{C}$ denote the space of compact subsets of $\Re^n$. The Hausdorff distance $\rho : \mathcal{C} \times \mathcal{C} \to \Re$ is defined by,

$$\rho(S, S') \equiv \left\{ \max \left[\max_{x \in S'} \min_{y \in S} \parallel x - y \parallel ; \max_{x \in S} \min_{y \in S'} \parallel x - y \parallel \right] \right\}$$

where $\parallel \bullet \parallel$ is the Euclidean norm.

Let $B_\epsilon(x) \equiv \{z \in \Re^n \mid \parallel x - z \parallel \, \leq \epsilon\}$ denote the *closed ϵ-ball around x*.

Let $int(S)$ denote the *interior* of S, and $\partial(S)$ the boundary of S.

Define the *weak Pareto frontier* of S as: $WP(S) \equiv \{x \in S \mid y \gg x \text{ implies } y \notin S\}$.

Define the *strong Pareto frontier* of S as: $P(S) \equiv \{x \in S \mid y \geq x \text{ implies } y \notin S\}$.

A bargaining problem (S, d) is said to be *strictly comprehensive* if $S = comp(S, d)$ and $WP(S) \equiv P(S)$. The domain of bargaining problems considered in this paper is Σ_c, the class of pairs (S, d) where $S \subset \Re^n$ and $d \in \Re^n$ such that:

A1) S is compact.

A2) S is d-comprehensive.

A3) There exists $x \in S$ and $x \gg d$.

This differs from the usual domain, which we denote Σ_{con}, in that we do not assume that the set of feasible utility allocations is convex. A *bargaining solution*, F, is a function from Σ_c to $\Re^n$ such that for each $(S, d) \in \Sigma_c$, $F(S, d) \in S$.

A list of axioms used in this paper follows:

Pareto Optimality (WPO): $F(S, d) \in WP(S)$.

Independence of Irrelevant Alternatives (IIA): If $S' \subset S$, $d' = d$, and $F(S, d) \in S'$, then $F(S', d') = F(S, d)$.

A *permutation operator*, π, is a bijection from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$. Π^n is the class of all such operators. Let $\pi(x) = (x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, \ldots, x_{\pi^{-1}(n)})$.[3] and $\pi(S) = \{y \in \Re^n \mid y = \pi(x)$ for some $x \in S\}$.

Symmetry (SYM): If for all permutation operators $\pi \in \Pi^n$, $\pi(S) = S$ and $\pi(d) = d$, then $F^i(S, d) = F^j(S, d) \; \forall \; i, j$.

An *affine transformation on $\Re^n$* is a map, $\lambda : \Re^n \to \Re^n$, where $\lambda(x) = a + bx$ for some $a \in \Re^n$, $b \in \Re^n_{++}$. Λ^n is the class of all such transformations. Let $\lambda(S) = \{y \in \Re^n \mid y = \lambda(x), x \in S\}$.

Scale Invariance (S.INV): $\forall \; \lambda \in \Lambda^n$, $F(\lambda(S), \lambda(d)) = \lambda(F(S, d))$.

Continuity (CONT): For all sequences of problems $\{(S^\nu, d)\}_{\nu=1}^\infty$, if $\rho(S, S^\nu) \to 0$, then $F(S^\nu, d) \to F(S, d)$.

The *Ideal Point* of a problem (S,d) is defined as:

$$a(S, d) \equiv (\max_{\substack{x \in S \\ x \geq d}} x^1, \max_{\substack{x \in S \\ x \geq d}} x^2, \ldots, \max_{\substack{x \in S \\ x \geq d}} x^n).$$

The next axiom was introduced in Roth (1980). It is a slight weakening of the *individual monotonicity* axiom introduced in Kalai and Smorodinsky (1975).

Restricted Monotonicity (R.MON): If $S \subset S'$, $d = d'$, and $a(S, d) = a(S', d')$, then $F(S', d') \geq F(S, d)$.

The *Ethical Point with respect to a solution F of a problem (S, d)*, is defined as:

$$E^F(S, d) \equiv F(con(S), d).$$

Ethical Monotonicity (E.MON): If $S' \subset S$, $d' = d$, and $E^F(S, d) \in con(S')$, then $F(S, d) \geq F(S', d')$.

Recall the motivation behind Restricted Monotonicity. We imagine an ideal world in which claims on the set of feasible allocations do not conflict. In such a world, we

would like to give all agents their highest possible level of utility. The vector made up of these maximal levels is called the "ideal point". However, in most interesting problems, claims do conflict and compromises must be made away from this target settlement. Restricted Monotonicity simply requires that if you take a bargaining problem and reduce the feasible set in a way that leaves the target settlement unchanged, then no agent should benefit from the decrease in opportunities. This captures the notion of "fairness," that all agents should share any such gains or losses.

Ethical Monotonicity is similar to Restricted Monotonicity. The only difference is in the way we identify a target settlement to the bargaining problem. We do this by first finding a solution concept that satisfies the list of axioms agreed upon by the agents. We imagine an ideal world in which agents could commit to accept the outcomes of lotteries or, more generally, a world in which we need only concern ourselves with the ex-ante allocations. We then take the point this solution concept recommends to the convex hull of the feasible set as our "ethical point". This is the settlement that is most desirable according to the ethical values summarized by the axioms agreed upon by the agents. We now follow the argument above in saying that if we reduce the feasible set in a way that leaves the target settlement unchanged, then no agent should benefit from the decrease in opportunities.

Note that Ethical Monotonicity in no way prohibits the settling of the problem at allocations attainable only through the use of lotteries. Where we break from standard practice is in our interpretation of Pareto optimality. We require that the solution outcome be an element of the weak Pareto frontier of the primitive feasible set. Implicit in the standard interpretation of Pareto optimality is that the solution point must lie on the Pareto frontier of the *convex hull* of the primitive feasible set. We have argued above that this is a nontrivial modeling choice. While this may be reasonable in many situations, for example when either binding contracts are available, or when only ex-ante fairness is important, there are many important situations in which the usual argument seems unrealistic. Therefore, the Pareto optimality axiom we use here is that the solution point be an element of the primitive feasible set.

4. The Nash Extension Solution

In his 1950 paper, Nash considered the domain Σ_{con} of convex problems. He proposed the following solution:

$$N(S,d) \equiv \left\{ \operatorname*{argmax}_{\substack{x \in S \\ x \geq d}} \prod_{i=1}^{n} (x_i - d_i) \right\},$$

and proved that it is the unique solution characterized by Pareto Optimality, Symmetry, Scale Invariance, and Independence of Irrelevant Alternatives. Kaneko (1980) offers a characterization of the direct generalization of Nash solution, the set of Nash product maximizers, on the domain of "regular" bargaining problems. Kaneko's solution is not single-valued and is only upper-hemicontinuous. Thus, his solution does not give an unambiguous recommendation of how to solve bargaining problems, and small changes in the problem can lead to large changes in the outcome. Herrero (1989) also defines a set-valued generalization of the Nash solution on the strictly comprehensive two person domain. Her solution, the set of "local" maximizers of the Nash product, satisfies a form of lower-hemicontinuity and selects a superset of the outcomes recommended by Kaneko's solution. Foster and Vohra (1988) give an elegant proof of Lensberg's characterization of the Nash solution on the domain of problems where the maximizer of the Nash product is unique.

One approach would be to define a new solution by taking a selection from the set of maximizers of the Nash product. Unfortunately, it is impossible to do this in a way that satisfies Nash's axioms. Obviously, no such selection satisfies Symmetry. In addition, *any* such selection must also fail to satisfy the axioms Continuity and Independence of Irrelevant Alternatives.

We wish to find a solution which is continuous, single-valued, coincident with the Nash solution if the problem is convex, and which approximates the Nash solution otherwise. The Nash Extension solution meets all these requirements. We construct the Nash Extension as follows. First define the mapping $L : \Sigma_c \to \Re^n$ as:

$$L(S,d) \equiv con\Big(N\big(con(S),d\big),d\Big).$$

$L(S, d)$ is the line segment connecting the disagreement point d to the Nash solution of the problem composed of the convex hull of S and d. Now we define the solution NE:

$$NE(S, d) \equiv \{\max x \mid x \in L(S, d) \cap S\},$$

[Insert Figure 1 here]

where max indicates the maximal element with respect to the partial order on $\Re^n$. The construction of NE is illustrated in Figure 1. The point $NE(S, d)$ is the intersection of the weak Pareto frontier of S and the line segment connecting the disagreement point and Ethical Point under the Nash solution to the problem (S, d). Obviously, NE coincides with N on the domain of convex problems. To see that the NE solution is single valued, notice that L is a nonempty, compact-valued correspondence. Then since L is also a line segment, its maximal element exists and is unique. Thus, NE is nonempty and single-valued on Σ_c. Lemma 1 shows that the NE solution is continuous.

Lemma 1. NE *is continuous on* Σ_c.

<u>Proof/</u>

Let $S^\nu \to S$. We begin by showing that con is a ρ-continuous correspondence. To see this, suppose that for any given $\epsilon > 0$, $\rho(S, S^\nu) \le \epsilon$. Then for any $y = \sum_{i=1}^{n+1} \alpha_i x_i \in con(S)$ there is a $y^\nu = \sum_{i=1}^{n+1} \alpha_i x_i^\nu \in con(S^\nu)$ such that $y \in B_\epsilon(y^\nu)$. By reversing the argument, we also find that for any $y^\nu \in con(S^\nu)$ there is a $y \in con(S)$ such that $y^\nu \in B_\epsilon(y)$. Thus if $\rho(S, S^\nu) \le \epsilon$ then $\rho(con(S), con(S^\nu)) \le \epsilon$, and so con is ρ-continuous.

Therefore, since con is continuous, and N is continuous on $\sum_{con}$, the composition map E^N, where $E^N(S, d) \equiv N(con(S), d)$, is continuous by Hildenbrand (1974), Proposition B.7. We conclude that if $S^\nu \to S$, then $E^N(S^\nu, d) \to E^N(S, d)$.

By definition, $NE(S, d) \in L(S, d)$. So $NE(S, d) = (1 - \lambda^*)d + \lambda^* E^N(S, d)$ for some $\lambda^* \in [0, 1]$. Also, for each S^ν, $NE(S^\nu, d) = (1 - \lambda^\nu)d + \lambda^\nu E^N(S^\nu, d)$ for some $\lambda^\nu \in [0, 1]$. Notice that the sequence $\{\lambda^\nu\}$ is drawn from the compact set $[0, 1]$. Thus, given any sequence of sets $\{S^\nu\}$ converging to S, if it can be shown for every convergent subsequence $\{\lambda^{\nu_k}\}$ of $\{\lambda^\nu\}$ that $\lambda^{\nu_k} \to \lambda^*$ then the lemma is proved. Suppose not. Then there are two cases:

1. Suppose first that for some subsequence $\{S^{\nu_k}\}$, $\lambda^{\nu_k} \to \hat\lambda$ and $\hat\lambda > \lambda^*$. Then the definition of NE implies $(1 - \hat\lambda)d + \hat\lambda E^N(S, d) \equiv \hat x \notin S$. Thus the sequence $\{NE(S^{\nu_k})\}$ converges to a point not in S, contradicting the hypothesis $S^{\nu_k} \to S$.

2. Now suppose that for some subsequence $\{S^{\nu_k}\}$ that $\lambda^{\nu_k} \to \hat\lambda$ and $\hat\lambda < \lambda^*$. Then $(1 - \hat\lambda)d + \hat\lambda E^N(S, d) \equiv \hat x \ll NE(S, d)$. Additionally, the existence of a point in S that strictly dominates d implies that $d \ll \hat x$. Hence by the d-comprehensiveness of S, $\hat x \in int(S)$. Thus there exists $\epsilon > 0$ and ν_1 such that for $\nu > \nu_1$, $B_\epsilon(\hat x) \subset S^{\nu_k}$ and $NE(S, d) \notin B_\epsilon(\hat x)$. Since $E^N(S^{\nu_k}, d) \to E^N(S, d)$, there exists ν_2 such that $\nu > \nu_2$ implies $L(S^{\nu_k}, d) \cap B_\epsilon(\hat x) \ne \emptyset$. Now, for each ν_k let $y^{\nu_k} = \max\{L(S^{\nu_k}, d) \cap B_\epsilon(\hat x)\}$. Let $\nu' \equiv \max\{\nu^1, \nu^2\}$. Clearly for $\nu_k > \nu'$, y^{ν_k} exists and $y^{\nu_k} \in \partial B_\epsilon(\hat x) \cap S^{\nu_k}$. However, by hypothesis $NE(S^{\nu_k}, d) \to \hat x$, so there exists ν'' such that $\nu_k > \nu''$ implies $NE(S^{\nu_k}, d) \in int(B_\epsilon(\hat x))$. Then for $\nu_k > \max\{\nu', \nu''\}$, we have $y^{\nu_k} \in$

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$S^{\nu_k} \cap L(S^{\nu_k}, d)$ and $y^{\nu_k} \gg NE(S^{\nu_k}, d)$, contradicting the definition of NE. Hence for every subsequence of S^ν, we have that $\hat\lambda = \lambda^*$. Therefore, $\hat{x} = NE(S, d)$.

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Our main result is a characterization of the new solution NE.

Theorem 1. *A solution on Σ_c satisfies Weak Pareto Optimality, Symmetry, Scale Invariance, Ethical Monotonicity, and Continuity if and only if it is the Nash Extension.*

<u>Proof/</u>

(a) First it is shown that the NE solution satisfies the axioms.

WPO: Let $x = NE(S, d)$. Assume there exists $y \in S$ such that $y \gg x$. Then since S is d-comprehensive there exists $z \in L(S, d) \cap S$ such that $z \gg x$. However, this contradicts the hypothesis $x = NE(S, d)$.

S.INV: Let $(S, d) \in \Sigma_c$ and $\lambda \in \Lambda^n$ be any affine transformation. Since $con(\lambda(S)) = \lambda(con(S))$, and N satisfies S.INV on Σ_{con}, we may therefore conclude that $N(\lambda(con(S)), \lambda(d)) = \lambda(N(con(S), d))$. This implies: $L(\lambda(S), \lambda(d)) = \lambda(L(S, d))$. Therefore, $\max\{L(\lambda(S), \lambda(d)) \bigcap \lambda(S)\} = \max\{\lambda(L(S, d) \cap S)\} = \lambda(NE(S, d))$, as required.

SYM: Let (S, d) be a symmetric problem. Then $(con(S), d)$ is also a symmetric problem. Since N satisfies SYM on Σ_{con}, $N(con(S), d)$ is a point of equal coordinates. But so is d, and so all elements $L(S, d)$ are points of equal coordinates. Consequently, $NE(S, d) \in L(S, d)$ is symmetric.

E.MON: Let $(S, d), (S', d')$ be such that; $S \subset S'$, $d = d'$ and $E^{NE}(S', d') \in con(S)$. Then $N(con(S'), d') = NE(con(S'), d') = E^{NE}(S', d')$, and therefore $N(con(S'), d') \in con(S)$. Since $con(S) \subset con(S')$, and N satisfies IIA on Σ_{con}, $N(con(S), d) = N(con(S'), d')$. Furthermore since $d = d'$ by hypothesis, $L(S, d) = L(S', d')$. Therefore $S \subset S'$ implies $NE(S, d) \le NE(S', d')$, as required.

CONT: See Lemma 1.

(b) Conversely let F be a solution on Σ_c satisfying the five axioms, and consider any problem (S, d). By S.INV, we can set $d = 0$ and $N(con(S), d) = (1, 1, \ldots, 1) \equiv I$.

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Then $NE(S,d) = (\alpha, \ldots, \alpha) \equiv x$ for some $\alpha > 0$. We distinguish two cases:

Case 1. $S \subset \Re^n_+$.

Let the sets T and V be defined as follows:

$$T \equiv concomp((n, 0, \ldots, 0), (0, n, \ldots, 0), (0, \ldots, n); d),$$

$$V \equiv T \setminus \{x + \Re^n_{++}\}.$$

Since $I = N(con(S), d)$, the hyperplane defined by $\sum_{i=1}^n x_i = n$ supports $con(S)$ at I. Hence $S \subset T$. Also, since F satisfies WPO, and S is comprehensive, $z \in \{x + \Re^n_{++}\}$ implies that $z \notin S$. Thus $S \subseteq V$.

Now, since $(V, 0)$ is a symmetric problem, and x is the only symmetric point in $WP(V)$, by WPO. and SYM, $F(V, d) = x$. Also, since I is the only symmetric point in $WP(T)$, by WPO. and SYM, $F(T, d) = I$. But $con(V) = T$, and so $E^F(V, d) = I$. Therefore, since $S \subseteq V$ and $E^F(V, d) = I \in con(S)$, by E.MON, $F(S, d) \leq F(V, d) = x$. There are two possibilities.

i) $x \in P(S)$. Then by WPO, $F(S, d) = x = NE(S, d)$ and the proof is complete.

ii) $x \notin P(S)$. Then consider the sequence of problems $\{(V^\nu; 0)\}$ and $\{(S^\nu; 0)\}$ defined by:

$$V^\nu \equiv \left\{V \bigcup comp\left[\frac{1}{\nu}I + (1 - \frac{1}{\nu})x; 0\right]\right\}$$

$$S^\nu \equiv \left\{S \bigcup comp\left[\frac{1}{\nu}I + (1 - \frac{1}{\nu})x; 0\right]\right\}.$$

Since V^ν is symmetric and $d = 0$, by WPO. and SYM, $F(V^\nu, 0) = (\alpha + \frac{1}{\nu}, \ldots, \alpha + \frac{1}{\nu}) \equiv x^\nu$. Since $S^\nu \subseteq V^\nu$, $E^F(V^\nu, 0) = I$ and $I \in con(S^\nu)$, by E.MON and WPO. we have $F(S^\nu, d) = F(V^\nu, d) = x^\nu$. But since $S^\nu \to S$, by CONT $F(S^\nu, d) \to F(S, d)$. Thus since $x^\nu \to x$, we conclude that $F(S, d,) = x = NE(S, d)$.

Case 2. $S \not\subset \Re^n_+$.

Let $V' \equiv \{V \bigcup_{\pi \in \Pi^n} \pi(S)\}$. Since V' is symmetric, and $x \in WP(V')$, we can replace $(V, 0)$ above with $(V', 0)$ and replicate the argument given for Case 1.

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The Nash Extension solution has two independently interesting properties. First, as we noted earlier, on the domain Σ_{con} it coincides with the Nash solution. In fact, on the convex domain, our new axiom, Ethical Monotonicity, along with Weak Pareto Optimality imply Independence of Irrelevant Alternatives.

Second, consider a domain of "anti-convex problems", Σ_{a-con}. This is the class of problems that satisfy $A1 - A3$ and the following two additional requirements.

A4) For all $x \in S, x \geq d$.

A5) $\{x \in \Re^n | x \geq d$ and $x \notin S\}$ is convex.

Anti-convex problems are "concave upward" instead of "concave down," and so might represent economies with increasing marginal returns instead of diminishing marginal returns. In this sense, convex and anti-convex economies are polar opposites. It is not hard to see that on this domain, Ethical Monotonicity, and Pareto Optimality imply Restricted Monotonicity. Recall the definition of the Kalai–Smorodinsky solution $K : \Sigma_c \to \Re^n$:

$$K(S, d) \equiv \max[x \in S \mid x \in con(a(S, d), d)].$$

We show in Conley and Wilkie (1991) that the Kalai-Smorodinsky solution is characterized by Weak pareto optimality, Symmetry, Scale Invariance, and Restricted Monotonicity, on Σ_c. Thus, on Σ_{a-con} the Nash Extension coincides with the Kalai-Smorodinsky solution. We show this formally in Theorem 2.

Theorem 2. *For all $(S, d) \in \Sigma_{a-con}, NE(S, d) = K(S, d)$.*

Proof/

Let $(S, d) \in \Sigma_{a-con}$. Apply an affine transformation $\lambda \in \Lambda$ to (S, d) such that $\lambda(d) = 0$, and $a(\lambda(S), 0) = (1, 1, \ldots, 1)$. Then $K(\lambda(S), 0)$ is the largest element of $\lambda(S)$ that is on the line segment connecting $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$.

However, by A4 and A5, $con(\lambda(S))$ just the unit simplex. Thus, $E^N(\lambda(S), 0) = N(con(\lambda(S), 0) = (1/n, 1/n, \ldots, 1/n)$. And so $NE(\lambda(S), 0)$ is the largest element on $\lambda(S)$ that is on the line segment connecting $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. Therefore, as both solutions satisfy S.INV, $NE(S, d) = K(S, d)$.

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Thus, the Nash Extension solution can be seen as a hybrid of the Nash and the Kalai-Smorodinsky solutions. The Nash Extension coincides with Nash in the ideal case in which all ex-ante feasible points can be achieved without resorting to lotteries, and with the Kalai-Smorodinsky solution when all ex-ante efficient and strictly individually rational allocations require that lotteries be used. Similarly, the axiom Ethical Monotonicity can be seen as a hybrid of Independence of Irrelevant Alternatives and Restricted Monotonicity.

5. Conclusion

Since Nash's pioneering treatment of the bargaining problem, most authors have maintained the assumption of convexity of the feasible set. Recently several authors have dispensed with the assumption, and investigated the properties of the known solutions. Anant, Mukherji and Basu (1990) and Conley and Wilkie (1991) show that the Kalai-Smorodinsky and egalitarian solutions remain well defined in Nash's sense, and that the original characterizations can be extended, if comprehensiveness is assumed. Extending the Nash solution, however, is not so straightforward. The approach adopted by Kaneko (1980), and Herrero (1989) is to allow set valued solutions and weaken the continuity requirement. Both authors provide characterizations of solutions that may be set-valued, but which coincide with the Nash solution under convexity. Herrero additionally is able to motivate her solution by its noncooperative implementation. The solution outcomes are exactly the (limit) stationary subgame perfect equilibria of the Rubinstein (1982) alternating offer game.

In this paper we proposed a new solution, the Nash Extension that retains several of the desirable features of the Nash solution. In particular, it is single-valued and continuous, on the domain of comprehensive problems, and coincides with the Nash

solution whenever the feasible set is convex. We provide a characterization of the new solution using Nash's original axioms except for Independence of Irrelevant alternatives, which we replace with a new axiom, Solution Monotonicity. Elsewhere, we show that it is possible to implement the solution in (limit) subgame perfect equilibria. We make the point that the literature has been too quick to assume that agents having von Neumann-Morgenstern utility functions means that it is safe to ignore nonconvex bargaining problems. Even under this utility assumption, there is still the important issue of whether ex-ante or ax-post satisfaction of the axioms is appropriate.

There are many directions for further research. It may be interesting to look at the properties of classes of solutions. It is easy to show that Thomson's (1986) characterization of the class of monotone path solutions extends to our domain. Another class of solutions that are well defined on the domain of convex problems is the class of strictly concave social welfare functions. The Nash solution is the most widely known of these. The class of solutions represented by an additively separable social welfare function has recently been axiomatized by Lensberg (1985). It would be of interest to see if our method of constructing the Nash Extension could be employed to define a new solution class on the domain of comprehensive problems, which coincides with the selection of the given class of functions, when the problem is convex. A characterization could then be attempted using Ethical Monotonicity or some similar axiom.

It may also be of interest to study more general domains. For example, suppose agents cannot necessarily dispose of utility freely, but they can "agree some of the time and disagree some of the time," then we have a domain of problems in which the feasible sets are star-shaped with respect to the disagreement point.

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