The Bargaining Problem Without Convexity:†
Extending the Egalitarian and Kalai-Smorodinsky Solutions.

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† **Published as:** John P. Conley and Simon Wilkie "The Bargaining Problem without Convexity: Extending the Egalitarian and Kalai-Smorodinski Solutions", *Economics Letters*, Vol. 36, 1991, pp. 365-9. We wish to thank William Thomson for advice and comments. All errors remain our own.

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Abstract

We relax the assumption used in axiomatic bargaining theory that the feasible set be convex. Instead we require only that it be comprehensive. We show that on this domain, Kalai's (1977) characterization of the Egalitarian solution remains true, as does Kalai and Smorodinsky's (1975) theorem if we use weak Pareto optimality.

1. Introduction

An *n-person bargaining problem* consists of a pair (S, d) where S is a non-empty subset of $\Re^n$, and $d \in S$. The set S is interpreted as the set of utility allocations that are attainable through joint action by all n agents. If the agents fail to reach an agreement, then the problem is settled at the point d, which is called the *disagreement point.* A *bargaining solution* F, defined on a class of problems Σ^n, is a map that associates with each problem $(S, d) \in \Sigma^n$ a unique point in S. In the axiomatic approach to bargaining we start by specifying a list of properties (Pareto-optimality, for example) that we would like a solution to satisfy. If it can be shown that there is a unique solution that satisfies a given list of axioms, then the solution is said to be *characterized* this list.

It is common to restrict the domain to problems with convex feasible sets. The standard justification for restricting attention to convex problems is an assumption that agents' preferences can be represented by von Neumann–Morgenstern utility functions, and then admitting the use of lotteries. We find this approach unappealing, for two reasons. First, the von Neuman-Morgenstern hypothesis is often rejected in empirical studies and several alternatives have been proposed, see Fishburn (1989) for a systematic exposition. Second, if lotteries are allowed the interpretation of the axioms becomes problematic, see Conley and Wilkie (1989). Recently several papers have adopted the axiomatic approach without the convexity assumption, see Anant *et al.* (1990), Herrero (1989), and Kaneko (1980).

In this paper we require only that the feasible set be comprehensive. This is equivalent to assuming free disposal in any underlying economic problem. Our results may be stated succinctly: (1) on our domain, there does not exist a solution that satisfies strong Pareto optimality and symmetry; (2) if we replace strong Pareto-optimality with weak Pareto-optimality, then Kalai and Smorodinsky's characterization of their solution on the domain of convex problems may be extended to the domain of comprehensive problems; and (3) Kalai's characterization of the egalitarian solution may be extended to the domain of comprehensive problems.

2. Definitions and Axioms

We start with some definitions and formal statements of the axioms used in the characterizations. Given a point $d \in \Re^n$, and a set $S \subset \Re^n$, we say S is *d-comprehensive* if $d \leq x \leq y$ and $y \in S$ implies $x \in S$.[1]

The *comprehensive hull of a set $S \subset \Re^n$, with respect to a point $d \in \Re^n$* is the smallest d-comprehensive set containing S:

$$comp(S; d) \equiv \{x \in \Re^n \mid x \in S \text{ or } \exists\, y \in S \text{ such that } d \leq x \leq y\}. \tag{1}$$

The *convex hull of a set $S \subset \Re^n$* is the smallest convex set containing the set S:

$$con(s) \equiv \left\{ x \in \Re^n \mid x = \sum_{i=1}^{n+1} \lambda_i y_i \text{ where } \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0\; \forall\, i, \text{ and } y_i \in S\; \forall\, i \right\}. \tag{2}$$

Define the *weak Pareto set of S* as:

$$WP(S) \equiv \{x \in S \mid y \gg x \text{ implies } y \notin S\}. \tag{3}$$

Define the *strong Pareto set of S* as:

$$P(S) \equiv \{x \in S \mid y \geq x \text{ implies } y \notin S\}. \tag{4}$$

The domain of bargaining problems considered in this paper is Σ_c^n. This is defined as the class of pairs (S, d) where $S \subset \Re^n$ and $d \in \Re^n$ such that:

A1) S is compact.

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[1] The vector inequalities are represented by $\geq, >$, and $\gg$.

A2) S is d-comprehensive.

A3) There exists $x \in S$ such that $x \gg d$.

The axioms used in this paper are:

Weak Pareto-Optimality (W.P.O.): $F(S,d) \in WP(S)$.

A *permutation operator*, π, is a bijection from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,n\}$. Π^n is the class of all such operators. Let $\pi(x) = (x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, \ldots, x_{\pi^{-1}(n)})$[2] and $\pi(S) = \{y \in \Re^n \mid y = \pi(x) x \in S\}$.

Symmetry (SYM): If for all permutation operators $\pi \in \Pi^n$, $\pi(S) = S$ and $\pi(d) = d$, then $F^i(S,d) = F^j(S,d)\ \forall\, i,j$.

An *affine transformation on* $\Re^n$ is a map, $\lambda : \Re^n \to \Re^n$, where for some $a \in \Re^n$ and $b \in \Re^n_{++}$, $\lambda(x) = a + bx$. Λ^n is the class of all such transformations. Let $\lambda(S) = \{y \in \Re^n \mid y = \lambda(x), x \in S\}$.

Scale Invariance (S.INV): $\forall\, \lambda \in \Lambda^n$, $F(\lambda(S), \lambda(d)) = \lambda(F(S,d))$.

Translation Invariance (T.INV): $\forall\, x \in \Re^n$, $F(S + \{x\}, d + x) = F(S,d) + x$.

Strong Monotonicity (S.MON): If $S \subset S'$ and $d = d'$, then $F(S',d') \geq F(S,d)$.

The *Ideal Point* of a problem (S,d) is defined as:

$$a(S,d) \equiv (\max_{\substack{x \in S \\ x \geq d}} x^1, \max_{\substack{x \in S \\ x \geq d}} x^2, \ldots, \max_{\substack{x \in S \\ x \geq d}} x^n). \tag{5}$$

Restricted Monotonicity (R.MON): If $S \subset S'$, $d = d'$, and $a(S,d) = a(S',d')$, then $F(S',d') \geq F(S,d)$.

3. The Results

First we show the impossibility result.

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[2] Subscripts indicate the components of a vector.

Theorem 1. *$\nexists\, F : \Sigma_c^n \to \Re^n$ such that F satisfies SYM and PO.*

Proof/

Consider the problem (S, d) where $S \equiv comp(\{(1,2) \bigcup (2,1)\}; (0,0))$ and $d \equiv (0,0)$. By PO, $F(S,d) = (2,1)$ or $F(S,d) = (1,2)$. But this contradicts SYM.

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Now we consider the Kalai-Smorodinsky solution, K:

$$K(S,d) \equiv \max\left[x \in S \mid x \in con(a(S,d), d)\right], \tag{6}$$

where max is with respect to the partial order on $\mathbf{R}^n$. The axioms used are equivalent to those used by Kalai and Smorodinsky (1975) to characterize K on the convex domain with two agents, except that only weak Pareto-optimality is used. For further discussion see Kalai and Smorodinsky (1975) and Thomson (1986).

Theorem 2. *A solution F on Σ_c^n satisfies SYM, S.INV, W.P.O, and R.MON if and only if it is the Kalai-Smorodinsky solution.*

Proof/

The proof that K satisfies the axioms is elementary and is omitted. Conversely let F be a solution satisfying the four axioms. Given any $(S,d) \in \Sigma_c^n$, assume by S.INV that the problem has been normalized such that $d = 0$ and $a(S,d) = (\beta, \dots, \beta) \equiv y$. Then $K(S,d) = (\alpha, \dots, \alpha) \equiv x$ for some $\alpha > 0$. Let T be defined as:

$$T \equiv comp(y; 0) \setminus \{x + \Re^n_{++}\} \tag{7}$$

and consider the problem $(T, 0)$. We distinguish two cases:

Case 1) $S \subset \Re^n_+$. Since S is comprehensive and $x \in WP(S)$, we have $S \subseteq T$. Also, since T is symmetric, $d = 0$, and x is the only symmetric element $WP(T)$, by W.P.O. and SYM, $F(T,0) = x$. However, since $S \subset T$, and $a(S,0) = a(T,0) = y$, by R.MON $F(S,0) \leq F(T,0) = x$

Now let T' be defined by,

$$T' \equiv comp[(\beta, 0, \ldots, 0), (0, \beta, \ldots, 0), \ldots, (0, \ldots, \beta), x; 0]. \tag{8}$$

Consider the problem $(T', 0)$. Since T is symmetric, $d = 0$, and x is the only symmetric element in $WP(T')$, then by W.P.O. and SYM, $F(T', 0) = x$. Also, since $T' \subset S$ and $a(S, d) = a(T', 0) = y$, by R.MON, $F(S, d) \geq F(T', d) = x$. Thus $F(S, d) = x = K(S, d)$.

Case ii) $S \not\subset \Re^n_+$. Let V be defined as follows,

$$V \equiv T \bigcup \left\{ \bigcup_{\pi \in \Pi} \pi(S) \right\}. \tag{9}$$

Note that V is symmetric and $S \subset V$. If we replace
$(T, 0)$ the previous argument with $(V, 0)$ the proof goes through as before.

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Last we examine the egalitarian solution, E,

$$E(S, d) \equiv \{\max [x \in S \mid x_i - d_i = x_j - d_j \; \forall \, i, j \in (1, \ldots n)]\}. \tag{10}$$

We show that Kalai's (1977) characterization of E is true on the comprehensive domain.

Theorem 3. *A solution F on Σ^n_c satisfies SYM, T.INV, W.P.O, and S.MON if and only if it is the egalitarian solution.*

Proof/

The proof that E satisfies the four axioms is elementary and is omitted. Conversely let F be a solution satisfying the four axioms. Given any $(S, d) \in \Sigma^n_c$, we can assume by T.INV that the problem has been normalized such that $d = 0$. Thus $E(S, d) = (\alpha, \ldots, \alpha) \equiv x$ for some $\alpha > 0$. Now let T be defined by:

$$T \equiv comp(x; 0), \tag{11}$$

and consider the problem $(T, 0)$. Since T is symmetric, $d = 0$, and x is the only symmetric element of $WP(T)$, by W.P.O. and SYM, $F(T, d) = x$. Also, since S is comprehensive $T \subseteq S$. Hence, by S.MON, $F(S, d) \geq x$.

By assumption, S is compact. Thus, there exists $\beta \in \Re$ such that $x \in S$ implies ($-\beta,\ -\beta,\ \ldots,\ -\beta) \leq (x^1, x^2, \ldots, x^n) \leq (\beta, \beta, \ldots, \beta)$. Let Z be the symmetric closed hypercube defined by:

$$Z \equiv \{y \in \Re^n \mid\ \forall\, i \mid y \mid\ \leq \beta\}. \tag{12}$$

Also define T' as:

$$T' \equiv Z \setminus \{x + \Re^n_{++}\}. \tag{13}$$

Consider the problem $(T'; 0)$. Since T' is symmetric, $d = 0$ and x is the only symmetric element of $WP(T')$, by W.P.O. and SYM, $F(T', d) = x$. But since $S \subseteq T'$, by S.MON, $F(S, d) \leq x$. Thus, $F(S, d) = x = E(S, d)$.

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4. Concluding Comments

In this paper we have examined the Kalai-Smorodinsky and Egalitarian bargaining solutions without the hypothesis that the feasible set is convex. We require only that the feasible set be comprehensive. We show that well known characterizations of these solutions extend to this domain. The comprehensive domain arises naturally from an assumption of free disposal on the underlying economic problem.

In a recent paper, Anant *et al* (1990) show that the Kalai-Smorodinsky theorem can be extended directly on the domain of two person "NE-Regular" problems where, below the ideal point the weak Pareto and Pareto sets coincide. Our first theorem shows that it is impossible to extend their result to the domain of comprehensive problems. However, since the set of comprehensive problems includes this class of NE-Regular problems, and the Kalai-Smorodinsky solution is always strongly Pareto-optimal on this class, our axioms imply strong Pareto-optimality on the domain of NE-Regular problems. Thus our second theorem implies the theorem of Anant *et al* (1990).

The work of Anant *et al* (1990) and this paper, suggests that the assumption of a convex feasible set is not essential for any Monotone Path Solution. Since any Monotone Path Solution is well-defined on the domain of comprehensive problems any characterization found on the domain of convex problems should be easy to adapt. This class of solutions is discussed and axiomatized Thomson (1986), pp 52-57. It is straightforward to extend this result to our domain.

The solution proposed by Nash (1950) is not well defined on our domain. We propose and characterize a new solution, which on the convex domain coincides with the Nash solution in a companion paper, Conley-Wilkie (1989). An alternative approach, allowing a solution to be a correspondence, is used in Kaneko (1980) and Herrero (1989).

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