Iterative Planning Procedures in Non-Convex and Informationally† Decentralized Economies

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Abstract

The price mechanism is extremely well suited to finding efficient allocations in convex market economies. However, if either tastes or production possibilities are not convex, then competitive equilibria may fail to exist. The price mechanism will not lead to an efficient outcome in this case. We must therefore look elsewhere for mechanisms to allocate resources in economies that exhibit such elementary and important non-convexities as externalities in production or consumption, and increasing returns to scale. In this paper a new mechanism is proposed in which the various firms in the economy are asked to report feasible points in the region of the goods space about which the central planner is the least informed. These reports are used to make simultaneous estimates of the firms' production sets from above and below. This contrasts with the approach used by Malinvaud(1967), Weitzman(1970), and others, in which the firms are questioned about production possibilities near the most preferred point of the current estimates of their feasible sets. In addition to being a Quantity–Quantity type algorithm as first discussed by Crémer(1977), the new procedure has the advantages of allowing systematic removal of entire regions of the production frontier from consideration as more information is gathered by the central planner, of providing an upper bound on the speed of convergence in order to facilitate comparison to other planning mechanisms, and of giving an estimate at every stage of how close the current plan is from an optimal one.

1. Introduction

The price mechanism is extremely well suited to finding efficient allocations in convex market economies. However, if either tastes or production possibilities are not convex, then competitive equilibria may fail to exist. The price mechanism will not lead to an efficient outcome in this case. We must therefore look elsewhere for mechanisms to allocate resources in economies that exhibit such elementary and important non-convexities as externalities in production or consumption, and increasing returns to scale. Many earlier writers on economic planning were motivated by an interest in socialist economies. It would be a mistake, however, for economists at large to ignore the area as a consequence of this history. The central message of the literature is that there exist mechanisms for making efficient resource allocation decisions other than the Walrasian price mechanism. This should be of general interest given that the Walrasian mechanism cannot be used in all economic situations.

The problem of economic planning is usually framed in terms of a central planning board (called the *planner* hereafter) charged with the task of finding a production plan that is optimal under a given social welfare function. Needless to say, this is only one of many possible interpretations. We could just as easily use such a mechanism to examine to problem of a centrally managed firm trying to allocate production over several different plants, or of a personnel manager trying to distribute various kinds of tasks to individual workers. In any event, the task is complicated by the generally maintained assumption that all the information about the production possibilities is privately held by the subsidiary economic units (call them firms). The planner solves the problem by using an iterative procedure in which questions about production possibilities are asked of each firm, and the answers used to name a tentative production plan and devise a new set of questions to ask the firms. The questioning continues until the tentative plan gets satisfactorily close to an optimal one.

The purpose of this paper is to present a new mechanism that represents an improvement over older ones in several respects. It builds on Crémer's (1977) quantity-quantity algorithm and shares the two main advantages of this kind of procedure. In

particular, firms need only be able to check whether or not a given commodity bundle can be produced. They are not required to know anything about the neighborhoods of feasible points (supporting hyperplanes, for example). Also, convexity of the production sets and the preference order is not required.

The main innovations are the way that the planner decides which regions of the production set to ask the firms questions about, and the way that the resulting information is used. The traditional approach is for the planner to somehow construct an estimate of the firms' production sets and then ask the firms to report feasible points near the most preferred part of the estimate. For example, the procedures of Crémer, Weitzman (1970), and Malinvaud (1967), all use this technique. In the new procedure, however, the firms are asked to report feasible points in whatever region the planner currently knows the least about. The planner comes to know a set of approximately evenly spaced, feasible points along each firms' production frontier as a result. This allows the him to construct a measure of his ignorance about the production possibilities that systematically decreases with each iteration. It also permits an upper bound on the speed of convergence to be calculated.

The novelty in information usage is based on the observation that it is possible to construct a lower bound as well as an upper bound on the production set using the feasible points reported to the planner by the firm. We then notice that any part of the overestimate that is inferior in the planner's preference ordering to any part of the underestimate cannot lie above an optimal point. Such regions may therefore be forgotten about and the planner's information refined only in the remaining areas of the production set. This is a significant advance over previous procedures which could not reject any part of the production frontier as non-optimal until convergence actually took place. The consequent ability to narrow the area of search speeds convergence, and reduces the memory requirements of the procedure. Most importantly, it allows the planner to estimate how close the tentative plan is to an optimal plan. Such a estimate is essential if the procedure does not converge in a finite number of iterations and so the planner has to make a decision about when to stop searching.

2. Planning Procedures

The purpose of this section is to give a general introduction to the planning literature and to show how the current paper fits in. Let us start by discussing the basic criteria by which planning procedures should be evaluated. Malinvaud, for example, suggests five properties that a good mechanism ought to have. (1) Well definedness: There never arises a situation in which the firm is faced with a question for which there is no truthful answer. (2) Feasibility: Each tentative production plan is feasible. (3) Monotonicity: Each successive tentative plan is better than the previous one. (4) Convergence in utility: As the number of iterations of the procedure increases without bound, the utility of the tentative plans tend to the utility of an optimal plan. (5) Finiteness: Convergence takes place in a finite number of iterations. These items are of varying importance. No one would dispute the necessity of well definedness or convergence in utility. Indeed, one might wish to strengthen the latter condition to require that the tentative plans converge to an actual optimal plan. Feasibility, and monotonicity are important if the procedure is likely to be stopped before convergence actually takes place. This will usually be the case since finite convergence can be expected only under special circumstances.

At any rate, this list certainly does not exhaust the set of desirable traits that a planning procedure might have. Consider the following: (6) Simple messages: The messages that are passed between planner and firm are "small" in some sense. The smaller messages, all else being equal, the less time consuming and error prone is the procedure likely to be. Since for all the mechanisms discussed in this paper, the messages are drawn from Euclidean spaces, "small" will be taken to mean that the messages have low dimension. This is not, of course, the only imaginable measure of simplicity. It has the advantage of being easy to quantify, and in any event, is desirable unto itself. (7) Generality: The procedure can be applied to a wide class of economic environments. This is important not only because it makes an algorithm more likely to be useful, but because it may be hard to check whether or not an economy really falls within a particular domain. The broader the domain, the smaller the chance of

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mistakenly using a procedure which does not apply, and for which convergence cannot be guaranteed. (8) Discrimination: The procedure identifies parts of the production frontier where optimal plans cannot lie and refrains from spending time searching there. The better a procedure is at this, the less likely it is to waste time proposing tentative plans far away from optimal plans. (9) Fast convergence: The procedure converges in as few iterations as possible. In practice, evaluating this means calculating an upper bound on the number of iterations it takes a procedure to start giving tentative plans that are no more than a specified distance from an optimal plan. Notice that this is stronger than merely calculating how long it takes a procedure to start giving tentative plans that are close to optimal in utility terms. The requirement is written this way in order to avoid the necessity of giving the planner's utility function any cardinal content.

Now let us consider several procedures in light of the criteria given above. Malinvaud and Weitzman each describe planning procedures that exploit the special features of convex economies. In Malinvaud's, the planner calls out a price vector at each iteration, and the firm[1] responds by sending the planner a profit maximizing output vector. The convex hull of these feasible points is taken as an estimate from below of the production set. Obviously, this estimate gets more accurate as more points are revealed by the firm.

In Weitzman's procedure, the planner asks the firm to produce the most preferred output vector on the current estimate of the production set. If the request is not feasible, the firm suggests a nearby feasible point as a compromise, and also gives the planner a supporting hyperplane to the production set at that point. By convexity, the upper halfspace of such a hyperplane cannot contain any feasible points. The intersection of all the lower halfspaces generated in this way may therefore be taken as an estimate from above of the true production set.

Both of these procedures are well defined, and the questioning is done in a way that guarantees convergence. Malinvaud's tentative plans are elements of an underes-

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[1] Only the one firm case is considered here. The procedures don't fundamentally change when more firms are added, but the ease of exposition is diminished.

timate of the production set that gets larger with each iteration. Consequently, the plans are both feasible and monotonic. Weitzman's tentative plans are elements of an overestimate of the feasible set which gets smaller with each iteration. As a result, his tentative plans are neither feasible nor monotonic.[2] In both cases, finite convergence can be expected only if the true production set is polyhedral. The only one of the other criteria that these two algorithms satisfy is that the messages be simple; they are just price and quantity vectors. Neither procedure can be successfully applied to non-convex economies, nor identifies irrelevant parts of the production frontier. Neither author supplies an estimate of the speed of convergence.[3]

Crémer proposes a procedure that addresses some of these problems. He works on a domain of economies that will be called *comprehensive* in this paper. By this is meant the class of economies for which preferences are monotonic, and production takes place under free disposal. In his algorithm, the planner first chooses a point known to lie above at least one optimal plan. The comprehensive hull of this point is taken as an initial estimate of the relevant part of the feasible set.[4] The planner then maximizes his utility over the estimate and demands that the firm produce a most preferred point. Obviously, this optimization takes place at the initial overestimate for the first iteration. If the point is not feasible, the firm offers as a compromise to produce some point on the boundary of the production set that is strictly dominated by the point the planner demands. By free disposal, the planner knows that all the points that dominate the compromise point are not in the feasible set. He therefore truncates the estimate of the feasible set by removing such points from the initial estimate. This is the natural analogue for comprehensive economies to Weitzman's technique of removing upper halfspaces as infeasible. The algorithm iterates as follows: the planner asks for

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[2] In fact, the procedure is inversely monotonic.

[3] It seems hard to imagine how one would go about constructing such an estimate. Both of these procedures search the goods space for an optimal plan in a very unsystematic way. This is also true for the Crémer procedure, which is discussed next.

[4] That is, the planner takes the space that lies below this first point as his initial overestimate of the feasible set. See expression refcomp for a formal definition of the comprehensive hull.

the most preferred point on the current overestimate of the production set. The firm responds with a compromise dominated by the demand. The planner then truncates the current estimate by eliminating all points strictly larger than the compromise and demands the new most preferred point. This continues until the planner demands a feasible point at which time, the procedure has obviously converged. The algorithm is illustrated in figure one.

Crémer's algorithm does well by the criteria on the list. It is well defined, although as in Weitzman's, the tentative plans are not feasible. Since the tentative plans are elements of an overestimate that contracts towards the true production frontier, it is also inversely monotonic. The procedure is convergent, and even in a finite number of iterations under certain restrictive assumptions. Like the other procedures, the message space is simple. The most important advantage of this algorithm over the first two is that it can be applied to a much more general class of economies. However, it does not discriminate, and Crémer does not give an estimate of the speed of convergence.

The procedure described in this paper is a hybrid of Crémer's and Malinvaud's. It is a Quantity–Quantity algorithm defined on the domain of comprehensive economies, and employs Crémer's method of constructing an overestimate of the production frontier. The message space is therefore small, being of the same dimension as the goods space, and the same large class of economies can be considered. The tentative plans, however, are taken from an underestimate that is constructed by taking the comprehensive hull of the feasible points reported by the firm. Thus, as in Malinvaud's procedure, the tentative plans are feasible, and converge monotonically to an optimal plan. In addition, it turns out to be possible to use the overestimate and underestimate together to discriminate against irrelevant parts of the production frontier. It will also be shown that the way that the planner decides which regions of the goods space to ask the firm questions about makes it possible to calculate an upper bound on the speed of convergence.

1a) Iteration 0

1b) Iteration 1

1c) Iteration 2

1d) Iteration 3

Figure 1: The Crémer procedure

3. A New Procedure

3.1 Definitions and Assumptions

This section lays out the basic definitions and assumptions that are used in the rest of the paper. First, consider the following notion of minimal "distance":[5]

$$\nu(A, B) \equiv \inf_{y \in B} \inf_{x \in A} \| x - y \| \tag{1}$$

where A and B are sets in $\Re^m$, and $\| \bullet \|$ denotes the Euclidean norm. ν gives the length of the smallest gap between two sets. Thus, if two sets, A and B, intersect, then $\nu(A, B) = 0$. A useful property of ν is given in lemma 1.

Lemma 1. *If $A' \subseteq A \subset \Re^m$ and $B' \subseteq B \subseteq \Re^m$, are all bounded then, $\nu(A', B') \geq \nu(A, B)$.*[6]

Proof/

By the definition of ν, and of compact sets, there must be two points, $a' \in$ closure(A'), and $b' \in$ closure(B') such that $\nu(A', B') \geq \nu(a', b')$. But then, $a' \in$ closure(A) and $b' \in$ closure(B). So $\nu(A', B') \geq \nu(a', b') \geq \nu(A, B)$.

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Let the planner's preference relation over the consumption set $X \subseteq \Re^m$ be symbolized by $\succ$ and $\succeq$ for strong and weak preference respectively. The weak upper and

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[5] Note that ν is not a measure of distance in a mathematical sense. In particular, it does not satisfy the triangle inequality.

[6] The three types of vector inequality will be symbolized by $\geq$ for the weak inequality between all elements of the vectors, $>$ for weak inequality between all elements of the vectors and strong inequality between at least one pair, and $\gg$ for strong inequality between all elements of the vectors. If $x \geq y$, then x is said to weakly dominate y. If $x \gg y$, then x is said to strongly dominate y.

lower contour sets of this relation for any subset $Z \subseteq X$ are denoted by $U(Z)$ and $L(Z)$ respectively. Formally:

$$U(Z) \equiv \{x \in X \mid x \succeq x' \text{ for some } x' \in Z\} \tag{2}$$

and,

$$L(Z) \equiv \{x \in X \mid x \preceq x' \text{ for some } x' \in Z\}. \tag{3}$$

The boundary of a set is denoted by "∂". Thus $\partial L(Z)$ is the boundary of the lower contour set of Z. Under the appropriate assumptions (like A5, A6, and A7) this is the indifference surface containing the least preferred point of Z (if Z is closed). The usual notion of set summation is used:

$$\sum_{i=1}^{t} Z^i \equiv \{x \in \Re^m \mid z = z^1 + z^2 + \cdots + z^t \text{ and } z^i \in Z^i \text{ for } i = 1, 2, \ldots, t\}. \tag{4}$$

This will be written $Z^1 + Z^2$ if only two sets are involved. The set of optimal production plans is called X^*:

$$X^* \equiv \{x \in Y \mid x \succeq y \ \forall \ y \in Y\}. \tag{5}$$

Two assumptions are made on each of the K firms' production sets $Y^\ell \subset \Re^m$:[7]

A1) Y^ℓ is closed for $\ell = 1, \ldots, K$.

A2) If $y \in Y^\ell$, then $y' \leq y$ implies $y' \in Y^\ell$ for $\ell = 1, \ldots, K$.

(Free disposal in production, or identically, comprehensiveness of Y^ℓ.)

Two assumptions are also made on the consumption set:

A3) X is closed.

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[7] Superscripts run from 1 to K, and index firms. The absence of a superscript indicates that a global object is being referred to. Note that making assumptions A1 and A2 for each firm results in the same two assumptions holding for the global production set $Y = \sum_{\ell=1}^{K} Y^\ell$.

A4) If $x \in X$ then $x' \geq x$ implies $x' \in X$.

(Adding more to a consumption vector does not take you out of the consumption set.)

Three assumptions are made on the planner's preferences:

A5) $\succeq$ is a complete and transitive order over X.

A6) For all $x \in X$, U(x), and L(x) are closed.

(Preferences are continuous.)

A7) If $x > x'$ then $x \succ x'$.

(Preferences are strongly monotonic.)

Finally, one assumption is made about the planner's information.

A8) For some $x \in X^*$, and $x^1 \in Y^1, x^2 \in Y^2, \ldots, x^K \in Y^K$, such that $\sum_{\ell=1}^{K} x^\ell = x$, the planner knows some:

(a) $b_0^1, b_0^2, \ldots, b_0^K$ such that $b_0^\ell \geq x^\ell$ for $\ell = 1, 2, \ldots, K$.

(b) $w_0^1, w_0^2, \ldots, w_0^K$ such that $w_0^\ell \leq x^\ell$ for $\ell = 1, 2, \ldots, K$.

Assumption 8 says that the planner knows an overestimate and an underestimate of at least one optimal plan, and also knows an overestimate and an underestimate of an element of each of the firms' production sets such that the sum of these elements equals this optimal plan. This is not a very strong assumption given that these initial estimates can be very far above or below the optimal plan.

Now consider the following definitions:

$$\Re_-^m \equiv \{x \in \Re^m \mid x \leq 0\}. \tag{6}$$

$$\{\Re_{++}^m\}^c \equiv \{x \in \Re^m \mid x \not\gg 0\}. \tag{7}$$

in words, $\Re_-^m$ is just the negative orthant, including the boundary, and $\{\Re_{++}^m\}^c$ is the *complement* of the strictly positive orthant. Notice that both are closed sets.

The next two correspondence are used to construct estimates of the production sets. For any set $Z \subseteq \Re^m$, define the *comprehensive hull of Z* as follows:

$$ch(Z) \equiv \bigcup_{x \in Z} (x + \Re^m_-) \tag{8}$$

This is the set of vectors in $\Re^m$ that are weakly dominated by some vector in Z. Now define the *inverse comprehensive hull of Z* to be:

$$ich(Z) \equiv \bigcap_{x \in Z} (x + \{\Re^m_{++}\}^c) \tag{9}$$

This is the set of vectors in $\Re^m$ that do not strictly dominate at least one vector in Z. Note that both the comprehensive hull and inverse comprehensive hull a finite collection of vectors is closed.

3.2 The Algorithm

The algorithm is based on the observation that it is possible to construct a lower bound as well as an upper bound on the production set of each firm based on the information acquired through the Crémer procedure. The planner is then able to use these two together to narrow the area of search for an optimal plan with each iteration. Basically, this is done by removing from consideration all parts of the overestimate of the production possibilities that are inferior to some part of the underestimate.

Crémer constructs an overestimate of the production set of each firm by using the fact that free disposal implies that any point that strictly dominates a point known to be on the production frontier cannot be feasible. Formally, he constructs the *Upper Estimate of the ℓth firm's production set at iteration n* thus:

$$UE^\ell_n \equiv ich(X^\ell_n) \bigcap ch(b^\ell_0) \tag{10}$$

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where X_n^ℓ is the set of points that firm ℓ has reported as being on its production frontier as of the n*th* iteration, and b_0^ℓ is the initial point that the planner demands of the firm as described in A8. UE_n^ℓ is a closed and comprehensive set as the intersection of two such sets.

Free disposal, however, can be applied in the "other direction" equally well. If a point x is known to be feasible, then all points that x dominates must also be feasible. The planner may therefore take the comprehensive hull of all known feasible points as a lower bound on the production set. Thus, the *Lower Estimate of the ℓth firm's production set at iteration n* is defined as:

$$LE_n^\ell \equiv ch(X_n^\ell). \tag{11}$$

In figure 2, the upper and lower estimates of a firm's production set are illustrated. Notice that the set theoretical difference between them is a union of "cubes".[8] This will also be true in higher dimension. Each cube may be uniquely characterized by a point b, which strictly dominates all other points in the cube (called the *best* point), and a point w, which is strictly dominated by all other points in the cube (called the *worst* point) via the correspondence:

$$\mathcal{C}(b, w) \equiv \{x \in \Re^m \mid b \geq x \geq w\}. \tag{12}$$

Notice now that the upper estimate can also be defined as the comprehensive hull of all the best points. That is:

$$UE_n \equiv ich(X_n^\ell) \bigcap ch(b_0^\ell) \equiv ch(B_n^\ell). \tag{13}$$

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[8] The use of the word "cube" is not meant to imply that these objects have equal sides, or that they are three dimensional. We use "cube" as a substitute for the more accurate, but awkward "hyper-rectangle".

$$X_n^\ell = \{x_1^\ell, x_2^\ell, x_3^\ell, x_4^\ell, x_5^\ell\}$$

$$B_n^\ell = \{b_1^\ell, b_2^\ell, b_3^\ell, b_4^\ell, b_5^\ell, b_6^\ell\}$$

$$W_n^\ell = \{w_1^\ell, w_2^\ell, w_3^\ell, w_4^\ell, w_5^\ell, w_6^\ell\}$$

$$C_n^\ell = \{(b_1^\ell, w_1^\ell); (b_2^\ell, w_2^\ell); (b_3^\ell, w_3^\ell); (b_4^\ell, w_4^\ell); (b_5^\ell, w_5^\ell); (b_6^\ell, w_6^\ell)\}$$

Figure 2

The algorithm itself involves the iterative updating of four sets of quantity vectors and the naming of a tentative production plan p_n^ℓ for each firm, for each iteration. The first of these is C_n^ℓ, a set of ordered pairs $\{(b_1^\ell, w_1^\ell), (b_2^\ell, w_2^\ell), \ldots, (b_i^\ell, w_i^\ell), \ldots, \} \equiv \{c_1^\ell, c_2^\ell, \ldots, c_i^\ell, \ldots, \}$ such that when these pairs are used to form cubes, their union equals the set theoretical difference between UE_n^ℓ and LE_n^ℓ. The next two are B_n^ℓ and W_n^ℓ, the sets of best and worst points in C_n^ℓ (that is, the set of all first and second elements, respectively, in the set of ordered pairs C_n^ℓ). It is notationally convenient to construct these sets separately even though all the information contained in them is also contained in C_n^ℓ. Last is X_n^ℓ, which is a set of points known to lie exactly on the firm's production frontier.

All of the sets above are constructed individually for each firm. The next set of definitions show how they are used to construct overestimates and underestimates of the global production frontier.

$$B_n \equiv \sum_{\ell=1}^{K} B_n^{\ell} \tag{14}$$

$$UE_n \equiv ch(B_n) \tag{15}$$

$$X_n \equiv \sum_{\ell=1}^{K} X_n^{\ell} \tag{16}$$

$$LE_n \equiv ch(X_n) \tag{17}$$

The algorithm is initiated as follows:

$$B_0^{\ell} \equiv \{b_0^{\ell}\} \text{ for } \ell = 1, \ldots, K \qquad B_0 \equiv \{b_0\} \equiv \left\{ \sum_{\ell=1}^{K} b_0^{\ell} \right\}$$

$$W_0^{\ell} \equiv \{w_0^{\ell}\} \text{ for } \ell = 1, \ldots, K$$

$$C_0^{\ell} \equiv \{c_0^{\ell}\} \equiv \{(b_0^{\ell}, w_0^{\ell})\} \text{ for } \ell = 1, \ldots, K \tag{18}$$

$$X_0^{\ell} \equiv \{w_0^{\ell}\} \text{ for } \ell = 1, \ldots, K \qquad X_0 \equiv \{w_0\} \equiv \left\{ \sum_{\ell=1}^{K} w_0^{\ell} \right\}$$

$$p_0^{\ell} \equiv \{w_0^{\ell}\} \text{ for } \ell = 1, \ldots, K \qquad p_0 \equiv \{w_0\} \equiv \left\{ \sum_{\ell=1}^{K} w_0^{\ell} \right\}$$

where b_0^{ℓ} and w_0^{ℓ} are overestimate and underestimate of the firm ℓ's contribution to an optimal plan (as described in A8).

The reader will notice that neither a set of global cubes nor a set of global worst points is collected. This is because the global best points are constructed in such a

way that their comprehensive hull gives an overestimate of the production set. The comprehensive hull of the known points X_n, on the other hand, gives an underestimate. These two estimates are all that are needed at the global level. Cubes are useful only in that their size is a measure of the planner's ignorance about a particular firm's production possibilities. The planner will end up asking the firms questions about the part of the production frontier contained in the "biggest" cube. Since no analogous question is ever asked at the global level, global cubes, and consequently, global worst points are not needed.

It will sometimes be necessary to recall exactly how a global object was constructed. The following convention is used. Each global object z (for example, a global best point or feasible point) is the sum of one element from each of the Z^ℓ's. That is, $z = z^1 + z^2 + \ldots + z^K$ where $z^\ell \in Z^\ell$ for $\ell = 1, \ldots, K$. Call this set of K elements that sum to z, $\bar{z}$.

$$\bar{z} \equiv (z^1, z^2, \ldots z^K) \in \Re^{m \times K}. \tag{19}$$

Clearly then, for the case of a set Z this becomes:

$$\bar{Z} \equiv \left\{ \begin{array}{c} \bar{z}_1 \\ \vdots \\ \bar{z}_i \\ \vdots \\ \bar{z}_s \end{array} \right\} \equiv \left\{ \begin{array}{cccc} z^1_{1_a}, & z^2_{1_b}, & \cdots & z^{K-1}_{1_c}, & z^K_{1_d} \\ \vdots & \vdots & & \vdots & \vdots \\ z^1_{i_a}, & z^2_{i_b}, & \cdots & z^{K-1}_{i_c}, & z^K_{i_d} \\ \vdots & \vdots & & \vdots & \vdots \\ z^1_{s_a}, & z^2_{s_b}, & \cdots & z^{K-1}_{s_c}, & z^K_{s_d} \end{array} \right\} \in \Re^{m \times K \times s}, \tag{20}$$

where $z^\ell_j \in Z^\ell$ for all j, and all $\ell = 1, \ldots, K$. We are finally ready to define the algorithm. Each iteration is broken down into five steps. Figure 3 gives an illustration for the one firm case.

Step 1) The first step for any given iteration n is to name a global tentative plan p_n. This is done by taking the most preferred point on the global lower estimate for the previous iteration, LE_{n-1}. Since $LE_{n-1} = ch(X_{n-1})$, and the planner's preferences are monotonic, this maximization must take place at some element of X_{n-1}. Each element

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3a)

3b)

3c)

3d)

Figure 3: The new procedure

$x \in X_{n-1}$ has a known decomposition $\bar{x} = (x^1, x^2, \ldots, x^K)$ where $x^\ell \in X_{n-1}^\ell \subset Y^\ell$ for $\ell = 1, \ldots, K$. Thus, p_n may be decomposed into $\bar{p}_n$, which in turn is taken as a specification of a tentative production plan for each firm. Formally:

1) Some $p_n \in \{x \in LE_{n-1} \mid x \succeq y \ \forall \ y \in LE_{n-1}\}$ is found and $\bar{p}_n$ is declared to be the tentative production plan for each firm.

Step 2) The second step is to find all elements of B_{n-1} that are strictly inferior to p_n, and discard them to form the new set $\tilde{B}_n$. We do this because any element of the overestimate is inferior to a feasible point p_n certainly cannot lie above an optimal plan. Such points may therefore be safely removed from future consideration. Likewise, elements that are known to be feasible, but which are nevertheless dominated by a discarded element of B_n may be thrown away. Formally:

2) $\tilde{B}_n \equiv \{b \in B_{n-1} \mid p_n \preceq b\}$
$\tilde{X}_n \equiv \left\{x \in X_{n-1} \mid b \geq x \text{ for some } b \in \tilde{B}_n\right\}$

Step 3) Having found and discarded the irrelevant elements of the global sets B_{n-1}, and X_{n-1}, the next step is to find the irrelevant elements of the sets collected from each of the firms. What makes an element of B_{n-1}^ℓ irrelevant? If some $b^\ell \in B_{n-1}^\ell$ is only used to create global best points that are known to be inferior to a feasible point (this is identical to saying that b^ℓ is only used to create global points that are not in the set $\tilde{B}_n$), then it can be concluded that the firm ℓ will never be called upon to produce b^ℓ, or any point that b^ℓ dominates. In other words, a best point may be discarded if no matter what the rest of the firms do, the point can never lie above a part of an optimal plan. Furthermore, all points of the sets W_{n-1}^ℓ and C_{n-1}^ℓ that are associated with discarded best points, and all elements of X_{n-1}^ℓ that are dominated by discarded points of B_{n-1}^ℓ may also be forgotten about. Formally:

3) $(\tilde{B}_n^1, \tilde{B}_n^2, \ldots, \tilde{B}_n^K) \equiv (\bar{\tilde{B}}_n)$
and for each $\ell = 1, 2, \ldots, K$:
$\tilde{W}_n^\ell \equiv \{w^\ell \in W_{n-1}^\ell \mid \exists \ (b^\ell, w^\ell) \equiv c^\ell \in C_{n-1}^\ell \text{ where } b^\ell \in \tilde{B}_n^\ell\}$
$\tilde{C}_n^\ell \equiv \{(b^\ell, w^\ell) \equiv c^\ell \in C_{n-1}^\ell \mid b^\ell \in \tilde{B}_n^\ell\}$
$\tilde{X}_n^\ell \equiv \{x^\ell \in X_{n-1}^\ell \mid b^\ell \geq x^\ell \text{ for some } b^\ell \in \tilde{B}_n^\ell\}.$

Step 4) Having discarded irrelevant information about each of the firms' production sets, the next step is to gain more information about relevant parts of them. In this algorithm the planner tries to get information about the part that he knows the least about. This is done in the following way. First notice that each cube has m edges radiating out from its best point. We are interested in the cube whose longest edge is longer than every other cube's longest edge. Let $(e^{1,\ell}, e^{2,\ell}, \ldots, e^{m,\ell})$ be the lengths of the m edges of the cube c^ℓ reorganized in descending order[9] (thus, $e^{1,\ell} \geq e^{2,\ell} \geq \ldots \geq e^{m,\ell}$). In all cases the first superscript refers to a component of a vector while the second refers to the firm. Let $E_n^{1,\ell}$ be the set whose elements are the lengths of the longest edges of each of the cubes in the set C_n^ℓ. That is to say, $E_n^{1,\ell} \equiv \{e_1^{1,\ell}, e_2^{1,\ell}, \ldots, e_i^{1,\ell}, \ldots\}$. Clearly, this set must have at least one maximal element $e_n^{*\ell}$, and there must be some cube c^ℓ whose longest edge has length $e^{1,\ell} = e_n^{*\ell}$. Then suppose that the longest edge of this cube is the j*th* edge. Let $h : \Re^m \times \Re^m \times \{1, 2, \ldots m\} \to \Re^m$ be defined as the hyperplane which is perpendicular to the j*th* edge and bisects the cube c:

$$h(b^\ell, w^\ell, j) \equiv \left\{ x \in \Re^m \mid x^j = \frac{(b^{j,\ell} + w^{j,\ell})}{2} \right\}. \tag{21}$$

The firm is asked to report some point on the production frontier within the cube c^ℓ, and on $h(b^\ell, w^\ell, j)$. That is to say, the firm is asked to report some point in the set:

$$PPF(b^\ell, w^\ell, j) \equiv \left\{ h(b^\ell, w^\ell, j) \cap \partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell) \right\}. \tag{22}$$

If such a point exists, it is added to the set $\tilde{X}_n^\ell$ to create X_n^ℓ. The set $PPF(b^\ell, w^\ell, j)$, however, may be empty. In that case, the production frontier must be either completely above or completely below the hyperplane within the cube. This is because of the assumption that Y^ℓ is closed, which implies that the production frontier is continuous. If either one of the above is true, the firm reports this instead of a point. In both cases, X_n^ℓ is put equal to $\tilde{X}_n^\ell$. It is now possible to conclude that the procedure is well defined since this is the only question that the firms are ever asked, and cases (a), (b), and (c), below, are exhaustive. Formally:

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[9] $(b^\ell - w^\ell)$ is the vector of these lengths.

4) For each $\ell = 1, 2, \ldots, K$, some cube $c^\ell \in \tilde{C}^\ell$ for which it is true that $e^{1,\ell} \geq e_i^{1,\ell}$ for all $e_i^{1,\ell} \in E_n^{1,\ell}$ is found. Let j be the longest edge of the cube c^ℓ. The firm is asked to report any point in the set $PPF(b^\ell, w^\ell, j)$. The firm makes one of three responses:

(a) $PPF(b^\ell, w^\ell, j) \neq \emptyset$, and the firm reports some element of the set, x_n^ℓ.

(b) $PPF(b^\ell, w^\ell, j) = \emptyset$ and $x < y \; \forall \; x \in \mathcal{C}(b^\ell, w^\ell) \cap h(b^\ell, w^\ell, j)$ and $y \in \partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell)$.

(c) $PPF(b^\ell, w^\ell, j) = \emptyset$ and $x > y \; \forall \; x \in \mathcal{C}(b^\ell, w^\ell) \cap h(b^\ell, w^\ell, j)$ and $y \in \partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell)$.

In case (a), $X_n^\ell \equiv \left\{ \tilde{X}_n^\ell \cup x_n^\ell \right\}$.

In case (b) and (c), $X_n^\ell \equiv \tilde{X}_n^\ell$.

Step 5) The fifth and final step of the algorithm, which is illustrated in figure 4, is to update the sets B_{n-1}^ℓ, W_{n-1}^ℓ, and C_{n-1}^ℓ. This was partially accomplished in step 2 when these sets were turned into $\tilde{B}_n^\ell$, etc. The updating is completed in different ways depending on what the firm reports at step 4. The first possibility is that the set $PPF(b^\ell, w^\ell, j)$ is not empty, and the firm reports some element x_n^ℓ to the planner. In this case, the cube c^ℓ is divided in two along the j axis, and c^ℓ is replaced with the two new cubes. Appropriate best and worst points are added to create B_n^ℓ and W_n^ℓ. The second possibility is that $PPF(b^\ell, w^\ell, j)$ is empty and $h(b^\ell, w^\ell, j) \cap \mathcal{C}(b^\ell, w^\ell)$ is below $\partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell)$. In this case c^ℓ is truncated by moving its worst point halfway up the longest edge of the cube. The last possibility is that $PPF(b^\ell, w^\ell, j)$ is empty and $h(b^\ell, w^\ell, j) \cap \mathcal{C}(b^\ell, w^\ell)$ is above $\partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell)$. Here, the opposite is done. The best point of the cube is moved halfway down the longest edge. Formally:

5) For each $\ell = 1, 2, \ldots, K$, if case (a) obtains:
$$B_n^\ell \equiv \{(\tilde{B}_n^\ell \setminus b^\ell) \cup b_i^\ell \cup b_j^\ell\}$$
$$\text{where } b_i^\ell \equiv b^\ell$$
$$\text{and } b_j^\ell \equiv (b^{1,\ell}, b^{2,\ell}, \ldots, b^{j-1,\ell}, \tfrac{b^{j,\ell}+w^{j,\ell}}{2}, b^{j+1,\ell}, \ldots, b^{m,\ell})$$
$$W_n^\ell \equiv \{(\tilde{W}_n^\ell \setminus w^\ell) \cup w_i^\ell \cup w_j^\ell\}$$
$$\text{where } w_i^\ell \equiv (w^{1,\ell}, w^{2,\ell}, \ldots w^{j-1,\ell}, \tfrac{b^{j,\ell}+w^{j,\ell}}{2}, w^{j+1,\ell}, \ldots, w^{m,\ell}),$$

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4a) In this example, assume that $e_n^* = e^1 = (b^1 - w^1)$. Then the cube c is to be divided according to case a) since $PPF(b, w, 1) \neq \emptyset$. As can be seen in the figure, c is divided exactly in half along the $1st$ edge creating two new cubes c_i and c_j, whose union equals c.

4b)   4c)

Figure 4: The three ways to divide a cube

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and $w_j^\ell \equiv w^\ell$

$$C_n^\ell \equiv \{(\tilde{C}_n^\ell \setminus c^\ell) \cup c_i^\ell \cup c_j^\ell\}$$

where $c_i^\ell \equiv (b_i^\ell, w_i^\ell)$ and $c_j^\ell \equiv (b_j^\ell, w_j^\ell)$

In case (b):

$$B_n^\ell \equiv \{(\tilde{B}_n^\ell \setminus b^\ell) \cup b_i^\ell\}$$

where $b_i^\ell \equiv b^\ell$

$$W_n^\ell \equiv \{(\tilde{W}_n^\ell \setminus w^\ell) \cup w_i^\ell\}$$

where $w_i^\ell \equiv (w^{1,\ell}, w^{2,\ell}, \ldots, w^{j-1,\ell}, \frac{b^{j,\ell}+w^{j,\ell}}{2}, w^{j+1,\ell}, \ldots, w^{m,\ell})$

$$C_n^\ell \equiv \{(\tilde{C}_n^\ell \setminus c^\ell) \cup c_i^\ell\}$$

where $c_i^\ell \equiv (b_i^\ell, w_i^\ell)$

In case (c):

$$B_n^\ell \equiv \{(\tilde{B}_n^\ell \setminus b^\ell) \cup b_i^\ell\}$$

where $b_i^\ell \equiv (b^{1,\ell}, b^{2,\ell}, \ldots, b^{j-1,\ell}, \frac{b^{j,\ell}+w^{j,\ell}}{2}, b^{j+1,\ell}, \ldots, b^{m,\ell})$

$$W_n^\ell \equiv \{(\tilde{W}_n^\ell \setminus w^\ell) \cup w_i^\ell\}$$

where $w_i^\ell \equiv w^\ell$

$$C_n^\ell \equiv \{(\tilde{C}_n^\ell \setminus c^\ell) \cup c_i^\ell\}$$

where $c_i^\ell \equiv (b_i^\ell, w_i^\ell)$

These five steps constitute one complete iteration of the algorithm. The next few lemmas substantiate the claim that the comprehensive hulls of B_n and X_n do indeed give upper and lower estimates of the production set.

Lemma 2. For $y \in \partial Y^\ell$, if $y \in \bigcup_{c \in \tilde{C}_n^\ell} \mathcal{C}(b, w)$, then $y \in \bigcup_{c \in C_n^\ell} \mathcal{C}(b, w)$.

Proof/

It must be shown that no part of the production frontier of a firm that is in some cube after step 3 is removed as a consequence of step 5. Since only one cube is altered by step 5, attention may be focused there. Step 5 can do three different things to the cube, c^ℓ, which is to be divided, depending on the circumstances. In case(a), c^ℓ is

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removed, and two cubes:

$$c_i^\ell \equiv [(b^{1,\ell}, \ldots, b^{m,\ell}); \ (w^{1,\ell}, w^{2,\ell}, \ldots w^{j-1,\ell}, \frac{b^{j,\ell} + w^{j,\ell}}{2}, w^{j+1,\ell}, \ldots, w^{m-1,\ell}, w^{m,\ell})]$$

and

$$c_j^\ell \equiv [(b^{1,\ell}, b^{2,\ell}, \ldots, b^{j-1,\ell}, \frac{b^{j,\ell} + w^{j,\ell}}{2}, b^{j+1,\ell}, \ldots, b^{m-1,\ell}, b^{m,\ell}); \ (w^{1,\ell}, \ldots, w^{m,\ell})]$$

are added to $\tilde{C}_n^\ell$ to form C_n^ℓ. But it is clear that $\mathcal{C}(b^\ell, w^\ell) = \mathcal{C}(b_i^\ell, w_i^\ell) \cup \mathcal{C}(b_j^\ell, w_j^\ell)$. Then trivially, since no $y \in \mathcal{C}(b^\ell, w^\ell)$ is removed, no $y \in \partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell)$ is removed.

In case(b), it need only be shown that the parts of ∂Y^ℓ inside the cube that is to be divided remain inside the resulting cube. Again, this is immediate since this could only be false if there were some $y^\ell \in \partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell)$, and $y^{j,\ell} < \frac{b^{j,\ell} + w^{j,\ell}}{2}$. But this is a violation of the conditions under which (b) is invoked. Similarly, in case (c), failure of the lemma implies for some $y^\ell \in \partial Y^\ell \cap \mathcal{C}(b^\ell, w^\ell)$, $y^{j,\ell} > \frac{b^{j,\ell} + w^{j,\ell}}{2}$, which violates the conditions of the case.

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Lemma 2 says that no part of the production frontier of a firm that is in some cube after step 3 is removed in step 5. Now consider the following definitions:

$$\tilde{C}_n \equiv \sum_{\ell=1}^{K} \tilde{C}_n^\ell \quad \text{and} \quad C_n \equiv \sum_{\ell=1}^{K} C_n^\ell. \tag{23}$$

These objects are the subject of the next two corollaries. Note that global cubes are not actually collected. Neither do they play any role in the definition of the procedure. They are briefly used here to help prove corollaries 2.1 through 2.3. Corollary 2.1 is just the global analogue of lemma 2.

Corollary 2.1 *For $y \in \partial Y$ if $y \in \bigcup_{c \in \tilde{C}_n} \mathcal{C}(b, w)$, then $y \in \bigcup_{c \in C_n} \mathcal{C}(b, w)$.*

Proof/

Immediate from lemma 2, and the definition of Y, $\tilde{C}_n$, and C_n.

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Corollary 2.2 *For $y \in \partial Y \bigcap \mathcal{C}(b_0, w_0)$, if $y \succeq p_n$, then $y \in \bigcup_{c \in C_n} \mathcal{C}(b, w)$.*

Proof/

This will be shown by induction. Since $\bigcup_{c \in C_0} \mathcal{C}(b, w) = \mathcal{C}(b_0, w_0) \equiv \mathcal{C}(\sum c_0^\ell)$ and $p_0 = w_0 \equiv \sum w_0^\ell$, for iteration 0 the statement reads: $\{y \in \partial Y \cap \mathcal{C}(b_0, w_0) \mid y \succeq w_0\} \subset \mathcal{C}(b_0, w_0)$. This is obviously true. Assume the statement is true for n. To show that it is true for n+1 it must be proven that the containment is preserved as C_n is changed into $\tilde{C}_{n+1}$, and as $\tilde{C}_{n+1}$ is changed into C_{n+1}. To see this for the first transition, take any $y \in \bigcup_{c \in C_n} \mathcal{C}(b, w)$ and suppose that $y \notin \bigcup_{c \in \tilde{C}_{n+1}} \mathcal{C}(b, w)$. By the induction hypothesis, there is some cube $c \in C_n$ such that $y \in \mathcal{C}(b, w)$ but $c \notin \tilde{C}_{n+1}$. But then, according to step 2, if c is removed from C_n then $b \prec p_{n+1}$. Since $y \leq b$ for any $y \in \partial Y \cap \mathcal{C}(b, w)$, it follows by monotonicity of preferences that $y \prec p_{n+1}$. To prove that the containment is preserved during the second transition, it is sufficient to show that for any $y \in \partial Y$, if $y \in \bigcup_{c \in \tilde{C}_{n+1}} \mathcal{C}(b, w)$, then $y \in \bigcup_{c \in C_{n+1}} \mathcal{C}(b, w)$. But this is immediate from corollary 2.1.

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Corollary 2.2 says that no feasible point on the production frontier is ever discarded if it is at least as good as the tentative plan.

Corollary 2.3 *For $y \in \partial Y \cap \mathcal{C}(b_0, w_0)$ if $y \succeq p_n$, then $y \in UE_n$.*

Proof/

Notice that $\bigcup_{c \in C_n} \mathcal{C}(b, w) \subset ch(B_n) \equiv UE_n$. Apply corollary 2.2.

∎

Corollary 2.3 states that the upper estimate at each iteration is indeed an overestimate of all the interesting parts of the production frontier.

Lemma 3. $LE_n \subset Y$.

Proof/

This follows directly from the definition of the LE as the comprehensive hull of feasible points and the assumption of free disposal.

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Lemma 3 shows that the lower estimate at any iteration n is as advertised.

Lemma 4. *For all iterations n, $W_n^\ell \subset LE_n^\ell$.*

<u>Proof/</u>

First we show by induction that for all n, and for all $c \in C_n^\ell$, there exists an $x \in X_n^\ell$ such that $x \in \mathcal{C}(b, w)$. The statement is true for $n = 0$ since c_0^ℓ is the only element of C_0^ℓ, $w_0^\ell \in X_0^\ell$, and obviously $w_0^\ell \in \mathcal{C}(b_0^\ell, w_0^\ell)$. See equations 18. Now suppose that the hypothesis is true for iteration n. Consider the following two classes of cubes:

1) First take the cubes $c \in C_{n+1}^\ell$ such that it is also the case that $c \in C_n^\ell$. By the induction hypothesis, there is some $x \in X_n$ such that $x \in \mathcal{C}(b, w)$. But since $b \in \tilde{B}_{n+1}$ and $b \geq x$, by step 3 of the algorithm, $x \in X_{n+1}$

2) Now consider cubes $\hat{c} \in C_{n+1}^\ell$ such that $\hat{c} \notin C_n^\ell$. This new cube has to have been the result of a division at step 5 of some cube $c \in C_n^\ell \subseteq \tilde{C}_{n+1}^\ell$. In case (a) of step 5, the firm reports a point $x \in h(b, w, j)$, and this point is added to X_{n+1}^ℓ. But the point x is in both of the cubes that result from division since it is in on their common boundary. In cases (b) and (c), by the induction hypothesis, there exists an $x \in X_n^\ell$ such that $x \in c$. But since $x \in \partial Y^\ell$, by the hypothesis of the case x must still be in the one cube, $\hat{c}$ that results

Thus, for all n, and for all $c \in C_n^\ell$, there exists an $x \in X_n^\ell$ such that $x \in \mathcal{C}(b, w)$. But then for all n, and for all $w \in W_n^\ell$ there exists an $x \in X_n^\ell$ such that $x \geq w$. Therefore, $W_n^\ell \subset ch(X_n^\ell) \equiv LE_n^\ell$.

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Lemma 4 is a technical fact which will be useful in proving future lemmas.

3.3 Technical Results on Cube Size

The purpose of this section is to prove some technical results. Lemmas 5–8, show that it is possible to find an upper bound of the length of any edge, of any cube, of any

firm, at every iteration n, and to show that this bound decreases as n goes to infinity in a predictable way. The superscripts that would normally signify firms are omitted to avoid excessive notation. It should be clearly understood that lemmas 5 through 8 apply to the size of cubes of any particular firm, and not to global cubes. Lemma 9 uses lemma 8 to show that there is an upper bound on the $\nu(L(p_n), U(X^*))$ at every iteration n. Readers uninterested in the details of how these bounds are calculated should skip directly to the next section.

Recall that the consequence of applying step five of the algorithm to a cube is its division into two cubes in case (a), or into one cube of half the original size in cases (b) and (c). We will call the cube or cubes that result from such a division *resultant cubes*. More generally, we will want to keep track of resultant cubes, cubes resulting from divisions of resultant cubes, and so on. The following notation will be used to indicate the pedigree of these classes of cubes. Consider the set of cubes C_n and let some unspecified number of iterations pass. Then the sets $C_n^r, C_n^{r^2}, C_n^{r^3}, \ldots$ will refer to the sets of cubes that are the result of a single division of a cube in C_n, two successive divisions, three successive divisions, etc. We may now state lemma 5.

Lemma 5. *Suppose a cube c_r results from the division of a cube c through step 5 at iteration n of the algorithm. If the jth edge of c is the edge to be divided then* $(b_r^j - w_r^j) = \frac{1}{2}(b^j - w^j) = \frac{1}{2}e^1 \leq \frac{1}{2}e_n^*$.

<u>Proof/</u>

Consider any cube c_r that results from a division along the j*th* edge of a cube c. The cube c_r must have resulted from application of one of cases (a), (b), or (c) in step 5. In case (a), c_r can take two forms:

$$\left[(b^1, b^2, \ldots, b^{j-1}, b^j, b^{j+1}, \ldots, b^m), (w^1, w^2, \ldots, w^{j-1}, \frac{b^j + w^j}{2}, w^{j+1}, \ldots, w^m) \right]$$

or

$$\left[(b^1, b^2, \ldots, b^{j-1}, \frac{b^j + w^j}{2}, b^{j+1}, \ldots, b^m), (w^1, w^2, \ldots, w^{j-1}, w^j, w^{j+1}, \ldots, w^m) \right].$$

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In case (b), only one form is possible:

$$\left[(b^1, b^2, \ldots, b^{j-1}, b^j, b^{j+1}, \ldots, b^m), (w^1, w^2, \ldots, w^{j-1}, \frac{b^j + w^j}{2}, w^{j+1}, \ldots, w^m) \right].$$

Likewise, in case (c), c_r must take the form:

$$\left[(b^1, b^2, \ldots, b^{j-1}, \frac{b^j + w^j}{2}, b^{j+1}, \ldots, b^m), (w^1, w^2, \ldots, w^{j-1}, w^j, w^{j+1}, \ldots, w^m) \right].$$

Then since either,

(i) $b_r^j - w_r^j = b^j - \frac{b^j + w^j}{2} = \frac{b^j - w^j}{2} = \frac{1}{2} e^1$

or

(ii) $b_r^j - w_r^j = \frac{b^j - w^j}{2} - w^j = \frac{b^j + w^j}{2} = \frac{1}{2} e^1$,

$(b_r^j - w_r^j) = \frac{1}{2}(b^j - w^j)$. But since a cube is always divided along it longest edge, and no edge of any cube can be longer that e_n^* at iteration n by definition, $(b_r^j - w_r^j) = \frac{1}{2}(b^j - w^j) = \frac{1}{2} e^1 \leq \frac{1}{2} e_n^*$.

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Lemma 6. *For any $c_{r^m} \in C_n^{r^m}$, $e_{r^m}^1 \leq \frac{1}{2} e_n^*$.*

Proof/

We start by distinguishing two exhaustive subclasses of cubes in $C_n^{r^m}$.

1. First consider cubes c_{r^m} in $C_n^{r^m}$ for which no edge has ever been subjected to two separate divisions. That is, cubes that are the result of m divisions of an original cube in C_n, and for which each of the m edges has been divided exactly once. By lemma 5, $(b_{r^m}^i - w_{r^m}^i) = \frac{1}{2}(b^i - w^i)$ for all $i = 1 \ldots m$. But by definition, $(b^i - w^i) \leq e^1 \leq e_n^*$ for all i. Thus, for cubes in this class, $e_{r^m}^1 \leq \frac{1}{2} e_n^*$.

2. Now consider cubes c_{r^m} in $C_n^{r^m}$ for which at least one edge has been subjected to at least two separate divisions. Without loss of generality, suppose that edge j is the longest edge of the original cube c, and so is the first to be divided. Then by lemma 5, $(b_r^j - w_r^j) = \frac{1}{2}(b^j - w^j)$. But by assumption, $\frac{1}{2}(b^j - w^j) \geq \frac{1}{2}(b^i - w^i)$ for all $i \neq j$. So if any edge is divided twice through

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step 5, then the jth edge is also divided twice. But since only the largest edge is ever subject to division, $\frac{1}{2}e_n^* \geq \frac{1}{2}e^1 = \frac{1}{2}(b^j - w^j) \geq (b_{r^m}^i - w_{r^m}^i)$ for all $i = 1 \ldots m$. Thus, for cubes in this class as well $e_{r^m}^1 \leq \frac{1}{2}e_n^*$. The lemma is proven.

∎

The point of lemma 6 is that if a cube is a result of m divisions of some original cube c, one of two things must be true: either each of the m edges of the resultant cube have been divided exactly once, or at least one of the edges has been divided twice (or more). In the former case, all the edges of the resultant cube are exactly half the length of those of c. So $e_{r^m}^1 = \frac{1}{2}e^1 \leq \frac{1}{2}e_n^*$ since $e^1 \leq e_n^*$ for all $c \in C_n$. In the latter case, if some edge is divided twice, then the jth edge must also have been divided twice (since the jth edge, being the longest edge, would be the first to be divided twice). But then half the length of the jth edge of the original cube must be longer than any other edge of the cube c_{r^m}, which proves the lemma.

Corollary 6.1 *For any $c \in C_n^{r^{m'}}$ for $m' \geq m, e^1 \leq \frac{1}{2}e_n^*$.*

Proof/

From lemma 6, $e^1 \leq \frac{1}{2}e_n^*$ for any $c \in C_n^{r^m}$. Since it is impossible for any edge of any cube to be increased as a result of a division, $e_{r^{m'}}^* \leq \frac{1}{2}e_n^*$ for $m' \geq m$.

∎

Corollary 6.1 generalizes lemma 6 to show that the conclusion holds for cubes that are the result of more than m divisions of an original cube.

Lemma 7. *If at some iteration n there are at most Q cubes in the set C_n, then after $Q(2^m - 1)$ more iterations, there will be at most $Q2^m$ cubes in the set $C_{[n+Q(2^m-1)]}$ and*

$$e_{[n+Q(2^m-1)]}^* \leq \frac{1}{2}e_n^*.$$

Proof/

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The first part is easy to show. Each iteration can add at most one cube. This is because the step 3 eliminates cubes, if it does anything at all, and (a) of step 5 adds one additional cube while (b) and (c) leave the number of cubes unchanged. Thus, an upper bound on the number of cubes at the end of iteration $n + Q(2^m - 1)$ when there were at most Q cubes at iteration n is $Q + Q(2^m - 1) = Q2^m$.

To see the second part, suppose initially that each of the cubes in C_n is divided once before any of the cubes is redivided. Then after Q iterations, $C_n \cap C_{[n+Q]} = \emptyset$. Also, $C_{[m+Q]}$ will consist of at most 2Q cubes. Now let each cube in C_{n+Q} be divided once before any is divided a second time. Since $C_{[n+Q]}$ contains at most 2Q cubes, this will take at most 2Q more iterations. At the end of these iterations $C_{[n+Q+2Q]}$ will contain at most $Q2^2$ cubes, and by construction, $C_n \cap C_{[n+Q+2Q]} = \emptyset$ and $C_n^r \cap C_{[n+Q+2Q]} = \emptyset$. Suppose that this process continues, and each cube in the set $C_n^{r^x}$ is divided before any cube in $C_n^{r^{x+1}}$ is. Then by the end of $n + Q + 2Q + Q2^2 + Q2^3 + \ldots + Q2^{m-1}$ iterations, there are at most $Q2^m$ cubes and

$$C_n^{r^x} \cap C_{[n+Q+2Q+Q2^2+Q2^3+,\ldots,+Q2^{m-1}]} = \emptyset$$

for all $x < m$. Pausing to show:

$$Q(1 + 2 + 2^2 +, \ldots, +2^{m-1}) = Q(2^m - 1)$$

$$(2 - 1)(1 + 2 + 2^2 +, \ldots, +2^{m-1}) = (2^m - 1)$$

$$(2 - 1 + 2^2 - 2 + 2^3 - 2^2 +, \ldots, +2^m - 2^{m-1}) = (2^m - 1)$$

$$2^m - 1 = 2^m - 1,$$

we may conclude that all elements of $C_{[n+Q(2^m-1)]}$ are elements of some $C_n^{r^{m'}}$ where $m' \geq m$. But by corollary 6.1, $e_{r^{m'}}^1 \leq e_{r^{m'}}^* \leq \frac{1}{2}e_n^*$ for any $c_{r^{m'}} \in C_n^{r^{m'}}$ for all $m' \geq m$. Therefore:

$$e_{[n+Q(2^m-1)]}^* \leq \frac{1}{2}e_n^*.$$

Now suppose that things do not develop so neatly and some cubes in $C_n^{r^{x+1}}$ are divided before some in $C_n^{r^x}$. Then assume some cubes have been divided less than m

times by the end of iteration $n + Q(2^m - 1)$ (or else corollary 6.1 can be applied directly as in the first case). Then since one cube must be divided at each iteration, if some cube is divided less than m times, some other cube must have been divided more than m times by the end of iteration $n + Q(2^m - 1)$. Thus at some stage, (say iteration n' where $n \leq n' \leq n + Q(2^m - 1)$), some $c \in C_n^{r^m}$ is divided. But this can only happen if $e^1 \geq e_j^1$ for all $c_j \in C_{n'}$. Then trivially, $e^1 \geq e_i^1$ for all $c_i \in C_{n+Q(2^m-1)}$. Since by corollary 6.1 $e^1 \leq \frac{1}{2} e_n^*$, we conclude:

$$\frac{1}{2} e_n^* \geq e^1 \geq e_{n+Q(2^m-1)}^*. \quad \bullet$$

Lemma 7 extends the argument of corollary 6.1 to say that if at iteration n, there are Q cubes, then after $Q(2^m - 1)$ more iterations, each of the original Q cubes must have been divided exactly m times, or at least one of the original cubes must have been divided more than m times. In both cases corollary 6.1 may be applied to conclude that the longest edge of any cube in $C_{n+Q(2^m-1)}$ is at most half the length of the longest edge of any cube in C_n.

Lemma 8. *At iteration* $I(t) = (2^m - 1) \sum_{s=0}^{t} 2^{sm}$, $e_{I(t)}^* \leq \frac{1}{2^{t+1}} e_0^*$.

Proof/

This is shown by induction. For t = 0, $I(t) = 2^m - 1$, so it must be shown that $e_{[2^m-1]}^* \leq \frac{1}{2} e_0^*$. At iteration $n = 0$, there are $Q = 1$ cubes in C_0, so by lemma 7, after $1 \times (2^m - 1)$ additional iterations,

$$e_{[0+(2^m-1)]}^* = e_{[(2^m-1)]}^* \leq \frac{1}{2} e_0^*.$$

Now assume that the statement is true for t. Then $e_{I(t)}^* \leq \frac{1}{2^{t+1}} e_0^*$. Since each iteration adds at most one cube, and there exists only one cube at iteration 0, after I(t) iterations, there are at most

$$I(t) + 1 =$$

$$(2^m - 1) \left(\sum_{s=0}^{t} 2^{sm} \right) + 1 =$$

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$$\left(\sum_{s=1}^{t+1} 2^{sm} - \sum_{s=0}^{t} 2^{sm} \right) + 1 =$$

$$2^{m(t+1)}$$

cubes in $C_{I(t)}$. Then by lemma 7, after at most $2^{m(t+1)}(2^m - 1)$ more iterations,

$$e^*_{[I(t)+2^{m(t+1)}(2^m-1)]} \leq \tfrac{1}{2} e^*_{I(t)}$$

$$\leq \frac{1}{2} \times \frac{1}{2^{t+1}} e^*_0$$

$$\leq \frac{1}{2^{t+2}} e^*_0$$

But

$$I(t) + 2^{m(t+1)}(2^m - 1) =$$

$$(2^m - 1) \left(\sum_{s=0}^{t} 2^{sm} \right) + (2^m - 1)2^{m(t+1)} =$$

$$(2^m - 1) \left(\sum_{s=0}^{t+1} 2^{sm} \right) =$$

$$I(t+1).$$

So at I(t+1), $e^*_{I(t+1)} \leq \frac{1}{2^{t+2}} e^*_0$.

■

Lemma 8 builds on lemma 7 to consider "blocks" of $Q(2^m - 1)$ iterations in order to calculate how many iterations must pass before e^*_n is smaller than $\frac{1}{2^t} e^*_0$.

The lemmas stated so far in this section have been about individual firms. Lemma 9 is about the global production set.

Lemma 9. *For $n \geq I(t)$, $\nu(L(p_n), U(X^*)) \leq \sum_{\ell=1}^{K} \frac{e_0^{\ell *} \sqrt{m}}{2^{t+1}}$*

<u>Proof/</u>

Consider any $x^* \in X^* \cap \mathcal{C}(b_0, w_0)$. By corollary 2.2, $x^* \in \mathcal{C}(b, w)$ for some $c \in C_n$. Then there exist some cubes $c^\ell = (b^\ell, w^\ell) \in C_n^\ell$ $\ell = 1, \ldots, K$ such that $\sum_{\ell=1}^{K} b^\ell = b \geq x^*$

and $\sum_{\ell=1}^{K} w^\ell = w \leq x^*$. But then there are $x^{*,\ell}$ such that $b^\ell \geq x^{*,\ell} \geq w^\ell$ and $\sum_{\ell=1}^{K} x^{*,\ell} = x^*$. Since by lemma 4, $w^\ell \in LE_n^\ell$, it follows that:

$$\nu(LE_n^\ell, x^{*,\ell}) \leq \parallel x^{*,\ell} - w^\ell \parallel \leq \parallel b^\ell - w^\ell \parallel$$

But by lemma 8, $b^{i,\ell} - w^{i,\ell} \leq \frac{e_0^{\ell*}}{2^{t+1}}$ for all $c^\ell \in C_n^\ell$, $i = 1, \ldots, m$, and $\ell = 1, \ldots, K$. Thus, recalling the definition of Euclidean distance:

$$\nu(LE_n^\ell, x^{*,\ell}) \leq \sqrt{\sum_{i=1}^{m} (b^{i,\ell} - w^{i,\ell})^2} \leq \sqrt{m \left(\frac{e_0^{\ell*}}{2^{t+1}}\right)^2} = \frac{e_0^{\ell*}\sqrt{m}}{2^{t+1}},$$

and so,

$$\nu(LE_n, x^*) \leq \sum_{\ell=1}^{K} \frac{e_0^{\ell*}\sqrt{m}}{2^{t+1}}.$$

But $x^* \in U(X^*)$. So by lemma 1, $\nu(LE_n, U(X^*)) \leq \sum_{\ell=1}^{K} \frac{e_0^{\ell*}\sqrt{m}}{2^{t+1}}$. Similarly, by step 1 of the algorithm, and lemma 3, $LE_n \subseteq L(p_n)$. Thus, by lemma 1 $\nu(L(p_n), U(X^*)) \leq \sum_{\ell=1}^{K} \frac{e_0^{\ell*}\sqrt{m}}{2^{t+1}}$.

■

Lemma 9 uses the limit given in lemma 8 on the length of any edge, of any cube in C_n^ℓ to find the maximum distance that any two points can be from one another provided both are in a given $c_i \in C_n$. It is then noted that $X^* \subset \bigcup_{c \in C_n} \mathcal{C}(b, w)$ for all n by corollary 2.2. Also, $p_n > w$ for all $w \in W_n$ Therefore, some cube in C_n contains both elements of $L(p_n)$ and $U(X^*)$. Thus an upper bound may be found for the minimum distance between these two sets for every iteration.

3.4 Results

At last we come to the results. Theorem 1 shows that the procedure is monotonic and gives feasible tentative plans.

Theorem 1. *The set of tentative production plans, $\{p_n\}_{n=1}^\infty$ are feasible, and monotonically increasing in the preference order.*

<u>Proof/</u>

$p_n \in LE_{n-1} \equiv ch(X_{n-1})$. By monotonicity of preferences, $p_n = x$ for some $x \in X_{n-1}$. But by construction, $x = \sum x^\ell$ for some $x^\ell \in Y^\ell$ for $\ell = 1, \ldots, K$. Thus $\bar{p}_n = \bar{x}$ and so $p_n^\ell = x^\ell \in Y^\ell$ for all $\ell = 1, 2, \ldots, K$. The tentative plans are therefore feasible.

Step 1 of the algorithm stipulates that $p_n \in \{x \in LE_{n-1} \mid x \succeq y \ \forall \ y \in LE_{n-1}\}$. But p_{n-1} is also an element of LE_{n-1} since only elements of LE_{n-2} that are strictly inferior to p_{n-1} are removed to form LE_{n-1}. It is immediate from the above that $p_n \succeq p_{n-1}$ for all n. Thus, $\{p_n\}_{n=1}^\infty$ is monotonically increasing in the preference order.

$\blacksquare$

Theorem 2 uses lemma 9 to show that the utility of the tentative plans converges to the utility of the optimal plans.

Theorem 2. *Given an economy satisfying A1-A8, the procedure converges in utility.*

<u>Proof/</u>

Assumptions A5, A6, and A7 are sufficient, according to Debreu(1954), to assure that $\succeq$ may be represented by a continuous utility function. Recall that by lemma 9, for $n \geq I(t)$, $\nu(L(p_n), U(X^*)) \leq \sum_{\ell=1}^K \frac{e_0^{\ell *} \sqrt{m}}{2^{t+1}}$. Thus, as $n \to \infty$, $\nu(L(p_n), U(X^*)) \to 0$. Then it is possible to choose a point $x^* \in X^*$ and a sequence $\{y^n\}$ such that $y^n \in L(p_n)$ for all n, and $\| y^n - x^* \| \to 0$. So for any particular continuous representation $u : \mathcal{C}(b_0, w_0) \to \Re$, $u(y^n) \to u(x^*)$. But since $u(y^n) \leq u(p_n) \leq u(x^*)$, $u(p_n) \to u(x^*)$.

$\blacksquare$

Theorem 3 strengthens theorem 2 to show that the tentative plans converge in quantity terms to actual optimal plans.

Theorem 3. *Given an economy satisfying A1-A8, the sequence of tentative plans $\{p_n\}$ converges to the set of optimal plans X^*.*

<u>Proof/</u>

Since $\{p_n\}$ is drawn from the compact set $\mathcal{C}(b_0, w_0)$, we need only show that the limit point of every convergent subsequence is an element of X^*. So take any convergent subsequence $\{p_n\}$ and suppose that $p_n \to p^*$. But by theorem 2, $u(p_n) \to u(x^*)$. Then by continuity $u(p^*) = u(x^*)$, and since by theorem 1 every tentative plan is a feasible element of $\mathcal{C}(b_0, w_0)$, $p^* \in X^*$.

■

Theorem 4 gives an upper bound on the speed of convergence in utility terms. Such an estimate is important because it gives the planner a basis to compare different planning procedures and decide which is best to solve his specific problem.

Recall from real analysis that all real continuous functions on a compact metric space are uniformly continuous. We will be forced to strengthen this somewhat in order to actually calculate a bound on the speed of convergence. In particular, we will assume that the utility function is *proportionally* uniformly continuous. In economic terms this means essentially that if two indifference curves are close to each other somewhere, then there is a proportionate bound on how far apart they can ever get from one another in the rest of the goods space. One interpretation of this is that marginal rates of substitution do not change very much in a small neighborhood.

A9) For any representation u of $\succeq$, $\exists\, \lambda > 0$ such that $\forall\, x \in \mathcal{C}(b_0, w_0)$, if $\|\, x - y\, \| \leq \epsilon$, then $|\, u(x) - u(y)\, | \leq \lambda\epsilon$.

(Proportional uniform continuity of utility.)

Theorem 4. *Given an economy satisfying A1-A9, for any $n \geq I(t)$, $u(x^*) - u(p_n) \leq \lambda \sum_{\ell=1}^{K} \frac{e_0^{\ell*}\sqrt{m}}{2^{t+1}}$.*

<u>Proof/</u>

By lemma 9 for $n \geq I(t)$, $\nu(L(p_n), U(X^*)) \leq \sum_{\ell=1}^{K} \frac{e_0^{\ell*}\sqrt{m}}{2^{t+1}}$. Thus, for every $n \geq I(t)$, there exist $x^n \in U(x^*)$ and $y^n \in L(p_n)$ such that $\|\, x^n - y^n\, \| \leq \sum_{\ell=1}^{K} \frac{e_0^{\ell*}\sqrt{m}}{2^{t+1}}$.

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But then by Assumption A9, and the fact the $x^n \succeq y^n$, $u(x^n) - u(y^n) \leq \lambda \sum_{\ell=1}^{K} \frac{e_0^{\ell*} \sqrt{m}}{2^{t+1}}$. Therefore, since $x^n \succeq x^*$ and $y^n \preceq p^n$, the theorem is proved.

∎

Unfortunately, it does not seem to be possible to extend theorem 4 and find a general bound on the speed of convergence in quantity terms. To do so we would have to know much more about the interactions between the preferences and the feasible set. For some subclasses of economies (convex economies for example), it may be possible to find useful characterizations of these interactions. But this will not be attempted in the current paper.

Finally, we turn to the question of stopping rules. Except in very special cases, finite convergence cannot be expected. So in practice, the planner will have abandon the search at some point and produce the current tentative plan. The planner must therefore devise a rule to stop the procedure when then the tentative plan is sufficiently "close" to an optimal plan. If the planner is satisfied with defining "closeness" in utility terms, stopping rules are very easy to implement. All he need do is subtract the utility of the tentative plan from the utility of the most preferred element of the Upper Estimate, and stop the search when this number falls below a pre-specified threshold. If the planner insists on making a stopping rule in quantity terms, then things are slightly more complicated. The planner must find an upper bound on the distance in goods space between the current tentative plan and the set of optimal plans. Recall that the algorithm discards information as it progresses. Then since we know that all $X^* \in \bigcup_{c \in C_n} \mathcal{C}(b, w)$, one way to find an upper bound is to take the maximum distance between the points in this union. Unfortunately, it will not always be the case that this bound goes to zero as the number of iterations goes to infinity. So a planner who uses a quantity stopping rule like this can never be sure that he will ever actually stop. Stopping is more likely, however, the closer set of optimal plans are to one another. In particular, if the set of optimal plans is a singleton, stopping is guaranteed.

4. Conclusions

In conclusion, the new procedure is well defined, monotonic, convergent, and even in a finite number of iterations, if the true production set is a step function.[10] In addition, The message space is simple, and the large class of comprehensive economies can be treated. Perhaps the most important advantage of the procedure presented in this paper is its ability to discriminate against irrelevant parts of the goods space and so to narrow the area of search. It is unfortunate that there is no obvious way to incorporate this fact into the estimate of speed of convergence. We also calculate upper bound on the number of iterations it takes the procedure to start suggesting tentative plans that are within any given ϵ in utility of the optimal plans. Although this is a very loose upper bound, it still can serve as a basis of comparison for speed of convergence with other procedures. Finally, we discuss ways to develop stopping rules for the new procedure. An interesting area for future research is investigating the conditions (if any) under which the procedure can converge but with a finite memory. That is to say, the conditions under which the sets B_n and so forth are finite for all n.

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[10] In Malinvaud's and Weitzman's procedure, finite convergence is obtained when the production set in polyhedral. This is because in this case the production set can be *exactly* approximated by the estimates that the procedure generates. The same thing holds here for step function production sets which can be exactly approximated by Quantity-Quantity type procedures. This is not a very important case, however, and no proof will be offered of this assertion.

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