Reference Functions and Possibility Theorems†
for Cardinal Social Choice Problems

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† **Published as**: John P. Conley, Richard McLean and Simon Wilkie, "Reference Functions and Possibility Theorems for Cardinal Social Choice Theory" *Social Choice and Welfare*, Vol. 14, 1997, pp. 65-78.

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Abstract

In this paper, we provide axiomatic foundations for social choice rules on a domain of convex and comprehensive social choice problems when agents have cardinal utility functions. We translate the axioms of three well known approaches in bargaining theory (Nash [1950], Kalai and Smorodinsky [1975], and Kalai [1977]) to the domain of social choice problems and provide an impossibility result for each. We then introduce the concept of a *reference function* which, for each social choice set, selects a point from which relative gains are measured. By restricting the invariance and comparison axioms so that they only apply to sets with the same reference point, we obtain characterizations of social choice rules that are natural analogues of the bargaining theory solutions.

1. Introduction

An *n-person social choice set* is a subset of $\Re^n$ representing the utility levels attainable by the ν agents through some joint action or decision. A *social choice rule* is a map defined on some domain of choice sets which for each choice set S selects a unique point in S. A *bargaining problem* consists of a pair (S, d), where S is a social choice set and $d \in S$ (the *disagreement point*) is interpreted as the allocation agents receive if they fail to reach an agreement. Given a class of bargaining problems, a *bargaining solution* is a map that associates with each problem (S, d) a unique point in S. We may view bargaining problems as a class of social welfare problems with additional structure. In the axiomatic approach to social choice and bargaining theory, one proposes a list of desirable axioms and attempts to show that they can be satisfied by one and only one allocation rule. In this case, the axioms are said to *characterize* the social choice rule or bargaining solution.

Despite the similarity between bargaining and social choice problems, the results in the literatures appear in sharp disagreement. In the bargaining literature, there are several well known solutions that have axiomatic characterizations. The solutions examined in Nash [1950], Kalai [1977] and Kalai and Smorodinsky [1978] are notable examples. Since Arrow's [1953] seminal paper, the most celebrated results in social choice theory have been impossibility theorems. One difference between these two frameworks is that bargaining theory typically assumes agents have von-Neumann Morgenstern utility functions. However, Sen [1970] showed that Arrow's impossibility result still holds even if one restricts the social choice domain to include only agents who have cardinal utility functions.[1] Hence, we must search elsewhere to find an explanation for the impossibility results in social choice theory.

Our main purpose in this paper is to investigate the reason for this contrast between the bargaining and social choice literatures. The approach we take begins by translating lists of axioms that characterize well known bargaining solutions to the domain of

[1] Also see Kalai and Schmeidler [1977] for a similar result.

convex and comprehensive social choice problems. This follows Sen [1970] and Myerson [1978], who showed that no social choice rule can satisfy the natural counterparts of the axioms that characterize the Nash bargaining solution. We continue this program by showing that there is no social choice rule that satisfies the social choice counterparts of either the sets of axioms that Kalai [1977] used to characterize the egalitarian solution, or those used by Kalai and Smorodinsky [1978] to characterize the solution proposed by Raiffa [1953].

Our main point is that this difference between possibility and impossibility may be driven not so much by the absence of a disagreement point in social choice problems, but more generally by the absence of *any* scale invariant point from which to measure relative utility gains. Without such a point, the cardinal nature of the utilities is ignored and the comparison axioms (such as Independence of Irrelevant Alternatives) are too strong. Motivated by this observation, we follow Thomson's [1982b] generalization of the bargaining theory framework and introduce the notion of a *reference function* to social choice problems. A reference function is simply a mapping from a class of social choice sets into utility space which satisfies certain properties. Restating the bargaining axioms on the domain of social choice problems which include reference functions now allows us to obtain possibility theorems.

The reader should note that there are two essential differences between a reference point and a disagreement point. First, a disagreement point must be specified as part of the data of a bargaining problem. The reference point on the other hand, is a function of the feasible set. Second, the disagreement point is interpreted as an allocation that any agent can unilaterally impose on the others. The reference point has no such interpretation, and may not even be in the feasible set. It is only a point from which relative gains are measured. In Conley, McLean, and Wilkie [1994], we extend this notion further to include the possibility of using a reference point to measure relative losses as well as relative gains. In this paper, we define a class of reference functions that allow us to characterize social choice rules that are analogous to the Nash, Raiffa and egalitarian, [1953] bargaining solutions.

Explorations of the boundary between possibility and impossibility have a long history in the social choice literature. We will not attempt to provide a comprehensive survey here, but we conclude this section with a brief discussion of some closely related papers.

There are two common approaches to regaining possibility results without allowing interpersonal comparability of utilities. One can restrict the class of allowable utility representations or one can limit the application of the independence axiom. For example, Plott and DeMeyer [1971] restrict the domain of preferences to those which are completely described by the "relative intensity" of preference over the alternatives. They then weaken the independence axiom and obtain a characterization of a class of social welfare functions. Note that the concept of relative intensity implies the use of a baseline allocation from which gains and losses are measured. This serves the same purpose as the reference point in this paper. More recently, Tsui and Weymark [1995] examine the existence of a social welfare ordering when the utilities are ratio scale measurable. They find that when negative utility values are allowed, if we require Pareto Optimality, then the social ordering must be dictatorial. However, when utilities are restricted to be positive (and thus zero can serve as a reference point) they obtain a characterization of the class of Cobb-Douglas social choice functions. Roberts [1980] began a systematic examination of how weakening the independence axiom compares with restricting the domain of allowable utilities. In particular, he introduced the notion of "partial independence of irrelevant alternatives," where the independence requirement is applied only to the union of subsets of the choice set and a particular fixed outcome. He established that if the utilities are scale invariant, then partial independence axiom and Pareto optimality characterize the class of weighted product of the player's utility differences from the fixed outcome (similar to Theorem 4 here).[2] D'Aspremont and Gevers [1987] provide characterizations of utilitarian and leximin type solutions in a social choice framework. They explore the ways in which

[2] See Roberts [1994] for an extension of these results.

invariance axioms involve comparisons of relative gains across agents.[3]

The approach we take in this paper is especially close to that of Gibbard, Hyland and Weymark [1987]. Their's is the first paper of which we are aware to explicitly examine the role of a fixed reference point to resolve the impossibility issue in the classical social choice context. The authors consider the standard Arrovian framework, but restrict the domain of choice sets to those for which a particular alternative is contained in all sets in the domain. Gains can then be measured from this common point. This weakening of the application of the independence axiom allows them to obtain a positive result. More recently, Dhillon and Mertens [1993] argue that the difference between possibility and impossibility is not whether the utility functions are cardinal or ordinal, but whether or not gains can be compared. They support this view by adapting Arrow's axioms for a domain of social choice problems where agents have von-Neumann Morgenstern utilities, and proving an impossibility result. They show, however, that by normalizing the agents' utilities and restricting the application of their independence axiom, it is possible to characterize the "relative utilitarian choice rule." Again, the normalization implies the existence of a "zero point" from which gains are measured. This literature supports our major thesis that the presence of a point from which to compare relative utility gains of agents is an aspect of the boundary between impossibility theorems and characterization results.

2. Definitions and Axioms

We start with some definitions and formal statements of the axioms used in the characterizations. Given a point $z \in \Re^n$ and a set $S \subset \Re^n$, we say S is *z-comprehensive* if $z \leq x \leq y$ and $y \in S$ implies $x \in S$.[4]

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[3] See our discussion of Thomson's [1981a] characterization of the utilitarian solution in section three for more details on this point.

[4] The vector inequalities are represented by $\geq, >, and \gg$.

The *comprehensive hull of a set* $S \subset \Re^n$*, with respect to a point* $z \in \Re^n$ is the smallest z-comprehensive set containing S:

$$comp(S; z) \equiv \{x \in \Re^n \mid x \in S \text{ or } \exists\, y \in S \text{ such that } z \leq x \leq y\}.$$

The *convex hull of a set* $S \subset \Re^n$ is the smallest convex set containing S. The *convex and comprehensive hull of a set* $S \subset \Re^n$ *with respect to a point* $z \in \Re^n$ is the smallest convex, z-comprehensive set containing S:

$$concomp(S; z) \equiv con(comp(S; z)).$$

Let $\mathcal{C}$ denote the space of non-empty compact subsets of $\Re^n$. The Hausdorff distance $\rho : \mathcal{C} \times \mathcal{C} \to \Re$ is defined by,

$$\rho(S, S') \equiv \max \left[\max_{x \in S'} \min_{y \in S} \parallel x - y \parallel, \max_{x \in S} \min_{y \in S'} \parallel x - y \parallel \right]$$

where $\parallel \bullet \parallel$ is the Euclidean norm. Let $int(S)$ denote the *interior* of S, and $\partial(S)$ the boundary of S. Define the *weak Pareto frontier* of S as:

$$WP(S) \equiv \{x \in S \mid y \gg x \text{ implies } y \notin S\}.$$

Define the *strong Pareto frontier* of S as:

$$P(S) \equiv \{x \in S \mid y \geq x \text{ implies } y \notin S\}.$$

We define the *ideal point* of S as

$$a(S) \equiv (\max_{x \in S} x_1, \max_{x \in S} x_2, \ldots, \max_{x \in S} x_n),$$

and the *nadir point* of S as

$$\nu(S) \equiv (\min_{x \in S} x_1, \min_{x \in S} x_2, \ldots, \min_{x \in S} x_n).$$

Note that neither the ideal point nor the nadir point need be a member of the social choice set in general.

We consider several domains of social choice problems in this paper. First, define the set of compact and convex social choice problems as follows:

$$\Sigma_{con} \equiv \{S \subset \Re^n \mid S \text{ is compact }, S = con(S) \text{ and } int(S) \neq \emptyset.\}\,.$$

The set of compact and nadir-comprehensive social choice problems is defined as:

$$\Sigma_{comp} \equiv \{S \subset \Re^n \mid S \text{ is compact }, S = comp(S; \nu(S)) \text{ and } int(S) \neq \emptyset.\}\,.$$

Of particular importance in this paper will be domain of compact, convex and nadir-comprehensive problems:

$$\Sigma_{cc} = \Sigma_{con} \cap \Sigma_{comp}.$$

Finally, the set of strictly nadir-comprehensive social choice problems is defined as:

$$\Sigma_{s.comp} \equiv \{S \in \Sigma_{comp} \mid P(S) = WP(S).\}\,.$$

A *social choice rule*, F, is a mapping from a class of problems Σ to $\Re^n$ such that for each $S \in \Sigma$, $F(S) \in S$.

Next, we define the class of affine transformations and permutation operators. We use the following convention: Given a and x in $\Re^n$ define $a \cdot x$ as the inner product and $ax \equiv (a_1 \cdot x_1, \ldots, a_n \cdot x_n) \in \Re^n$.

A *positive affine transformation* $\lambda = (a, b) \in \Re^n_{++} \times \Re^n$ maps $x \in \Re^n$ to $ax + b \in \Re^n$ so that $\lambda(S) \equiv \{y \in \Re^n \mid y = ax + b, \text{ and } x \in S\}$ for each $S \subset \Re^n$.

We will use the notation $\mathbf{1}$ for the vector $(1, \ldots, 1)$.

A *permutation operator*, π, is a bijection from $\{1, \ldots, n\}$ onto $\{1, \ldots, n\}$ and Π^n denotes the class of all such operators. For each $\pi \in \Pi^n$, we abuse notation and define $\pi(x) = (x_{\pi^{-1}(1)}, x_{\pi^{-1}(2)}, \ldots, x_{\pi^{-1}(n)})$ and $\pi(S) = \{y \in \Re^n \mid y = \pi(x) \text{ and } x \in S\}$.

A social choice set S is said to be *symmetric* if for all $\pi \in \Pi^n$, $S = \pi(S)$.

We close this section by defining social choice equivalents of several axioms which are commonly used in bargaining theory. Since social choice problems do not include disagreement points as part of the data, this translation is accomplished by omitting any reference to disagreement points.

Weak Pareto Optimality (WPO): $F(S) \in WP(S)$.
Pareto Optimality (PO): $F(S) \in P(S)$.
Symmetry (SYM): If for all $\pi \in \Pi^n$, $\pi(S) = S$, then if $x = F(S)$, $x_i = x_j$ for all i, j.
Translation Invariance (T.INV): For all $x \in \Re^n$, $F(S + x) = F(S) + x$.[5]
Homogeneity (HOM): For any $a \in \Re^n_{++}$ such that $a_i = a_j$ for all i, j, $F(aS) = aF(S)$.
Scale Invariance (S.INV): For all affine transformations $(a, b) \in \Re^n_{++} \times \Re^n$, $F(aS+b) = aF(S) + b$.
Continuity (CONT): If
$\{S^\nu\}_{\nu=1}^{\infty}$ is a sequence of problems, then $\rho(S, S^\nu) \to 0$ implies $F(S^\nu) \to F(S)$.
Strong Monotonicity (S.MON): If $S \subset S'$, then $F(S') \geq F(S)$.
Restricted Monotonicity (R.MON): If $S \subset S'$ and $a(S) = a(S')$ then $F(S) \leq F(S)$.
Independence of Irrelevant Alternatives (IIA): If $S \subset S'$ and $F(S') \in S$, then $F(S) = F(S')$.

3. The Impossibility Results

We prove three impossibility theorems in this section. We begin by considering Nash's [1950] solution for the domain of convex bargaining problems:

$$N(S, d) \equiv \underset{\substack{x \in S \\ x \geq d}}{\text{argmax}} \prod_{i=1}^{n} (x_i - d_i),$$

[5] Throughout the paper, we will not distinguish between a point $x \in \Re^n$ and a singleton set $\{x\}$.

Nash proves that this is the only solution that satisfies the bargaining theory analogues of WPO, SYM, S.INV, and IIA. Sen [1970] and Myerson [1978], however, proved that there is no social choice rule that satisfies WPO, SYM, S.INV, and IIA on the domain of convex problems. We include the following proof for completeness.

Theorem 1. *On Σ_{cc} there is no social choice rule that satisfies PO, SYM, HOM, T.INV, and IIA.*

Proof/

Let $n = 2$ and $S = con((1,0),(0,1),(0,0))$. By SYM and PO, $F(S) = (\frac{1}{2}, \frac{1}{2})$. Let $a = (2,2)$ and define $S' = aS$. By HOM, $F(S') = (1,1)$. Now define $S'' = S + (0,1)$. By T.INV, $F(S'') = (\frac{1}{2}, 1\frac{1}{2})$. However,

$$F(S') = (1,1) \in S'',$$

and $S'' \subset S'$. Thus, by IIA, $F(S'') = (1,1) \neq (\frac{1}{2}, 1\frac{1}{2})$, a contradiction.

■

Remark 1.1. S.INV implies HOM and T.INV. Thus, on our domain, Theorem 1 implies Sen's result.

Remark 1.2. The comprehensiveness assumption is not required for this result.

Remark 1.3. In Conley, McLean, and Wilkie [1994] we define a "dual" scale invariance axiom and use it to characterize the minimum Euclidean distance solution advocated by Yu [1973]. This axiom implies both HOM and T.INV. Thus, Theorem 1 implies that there is no choice rule that satisfies the social choice counterparts of the axioms used in that paper.

Thomson [1981a] proves the utilitarian choice rule, $u(S) = argmax_{x \in S} \sum x_i$, satisfies PO, SYM, T.INV, and IIA, and is the only such choice rule. The utilitarian choice rule also satisfies homogeneity, and hence Thomson's result seems to violate the above theorem. However, Thomson's theorem holds on the domain of strictly convex feasible sets. On the domain of convex choice sets, the set of maximizers of the function u above may not be single valued and hence the utilitarian choice rule is not well defined.

Thomson suggests that the utilitarian rule may be interpreted as choosing the same allocation as the Nash bargaining solution "in the limit" where the disagreement point is $(-k, \ldots, -k)$ as k approaches infinity. See Thomson [1981a] for further discussion.

Moulin [1988, Theorem 2.3], proves that, on the space $\Re^n{}_{++}$, the Nash social welfare ordering is characterized by axioms similar to those above, apart from T.INV. If we confine all choice sets to lie within the positive orthant, then a social choice rule can be defined by selecting the maximal element in the choice set with respect to the Nash social welfare ordering. However, this result cannot be extended to $\Re^n$. Indeed, Moulin shows that his characterization is equivalent to Nash's bargaining theorem where the disagreement point has been normalized to zero. In the absence of such a prespecified point from which to measure utility gains, the approach described above fails.

Kaneko and Nakamura [1975] consider a slightly different domain. They consider an environment in which there is a convex set of pure alternatives, X, and a particular least desirable alternative x_0. They then define the set of mixed alternatives M as the set of lotteries over $X \cup x_0$. They show that the modified Nash social welfare function,

$$w(S) = argmax_{m \in M} \sum ln(u_i(m) - u_i(x_0))$$

is the only solution to satisfy versions of weak Pareto optimality, anonymity, continuity, and IIA. appropriately modified for their domain of problems. They obtain a positive result because they assume the existence of a least desirable point. Again, this allows them to use the image of this point in utility space as an origin from which to measure utility gains.

We now consider the solution proposed by Raiffa [1953], and characterized by Kalai and Smorodinsky [1975] with bargaining versions axioms of SYM, S.INV, WPO, and R.MON. This solution, denoted R, is defined as follows:

$$R(S,d) \equiv t^* d + (1-t^*)a(S), \text{ where } t^* = min\{t \in \Re_+ \mid td + (1-t)a(S) \in S\}.$$

We show below that no solution can satisfy the counterparts of these axioms on the domain of comprehensive social choice problems.

Theorem 2. *On Σ_{cc} there is no social choice rule that satisfies WPO, SYM, S.INV, and R.MON.*

Proof/

Let $n = 2$ and $S = comp(con((0,4),(3,3),(4,0));(-4,-4))$. By SYM and WPO, $F(S) = (3,3) \equiv x$. Let $T = con((2,-2),(-2,2),(1\frac{1}{6},1\frac{1}{6}))$. By SYM and WPO, $F(T) = (1\frac{1}{6},1\frac{1}{6}) \equiv y$. Let $U = aT + b$ where $a = (1,2)$ and $b = (2,0)$. Thus, $U = con((4,-4),(0,4),(3\frac{1}{6},2\frac{1}{3}))$. By S.INV, $F(U) = ay + b = (3\frac{1}{6},2\frac{1}{3})$. However, $U \subset S$ and $a(U) = a(S) = (4,4)$, and therefore R.MON implies $F(U) \leq (3,3)$. Since $(3\frac{1}{6},2\frac{1}{3}) \not\leq (3,3)$, we have a contradiction.

■

Remark 2.1. The convexity assumption is not required for the above result.

Kalai [1977] examines the properties of an alternative bargaining solution. He defines the *egalitarian solution*, E, as

$$E(S,d) \equiv d + (t^*)\mathbf{1}, \text{ where } t^* = max\{t \in \Re_+ \mid d + t\mathbf{1} \in S\},$$

and shows that this is the only solution satisfying the bargaining counterparts of the axioms SYM, T.INV, WPO, HOM, and S.MON. Note that the egalitarian solution cannot satisfy scale invariance since gains must be shared equally. Equal division of gains implies a common scale on which such gains can be weighed. On the other hand, egalitarian solutions are translation invariant since, when both the disagreement point and the feasible set are moved by the same amount, equal division of gains results in the same final allocation. We now show that no choice rule can satisfy the counterparts of these axioms on the domain of convex and comprehensive social choice problems.

Theorem 3. *On Σ_{cc} there is no social choice rule which satisfies WPO, SYM, T.INV, and S.MON.*

Proof/

Let $n = 2$, $S = con((2,0),(0,2),(0,0))$, and $T = con((1,0),(0,1),(0,0))$. By SYM and WPO, $F(S) = (1,1)$ and $F(T) = (\frac{1}{2}, \frac{1}{2})$. Let $U = T + (0,1)$. By T.INV, $F(U) = (\frac{1}{2}, 1\frac{1}{2})$. However $U \subset S$ and therefore by S.MON, $F(U) \leq (1,1)$. a contradiction.

■

Remark 3.1. The convexity assumption is not required for the above result.

4. Possibility with Reference Points

In the previous section, we showed that it is impossible for social choice rules to satisfy the sets of axioms most commonly used in bargaining theory. The difference between social choice and bargaining problems is the absence of a disagreement point in the former. In this section, we argue more generally that it is the absence of a reference point from which to measure gains, of which the disagreement point is just a special case, which drives these impossibility results. We illustrate this by providing a characterization of Nash, egalitarian and Raiffa social choice rules using axioms that include reference functions in their statements. We conclude that positive results are possible on the social choice domain if we have reference points satisfying certain requirements.

We begin with some technical details. Define Δ to be the line of points of equal coordinates :

$$\Delta \equiv \{x \in \Re^n | x_i = x_j \ \forall \ i, j\}.$$

Given a set S, let $\bar{S}$ be the smallest symmetric set in Σ_{cc} containing S in the sense of set inclusion, and let $\underline{S}$ be the largest symmetric set in Σ_{cc} contained in S. Note that when $S \in \Sigma_{cc}$, $\bar{S}$ will always exist, but $\underline{S}$ need not exist. It is straightforward to show that $\bar{S} = concomp\{\cup_{\pi \in \Pi^n} \pi(S)\}$, and $\underline{S} = \cap_{\pi \in \Pi^n} \pi(S)$.

Next, we follow Thomson [1981b] and define a class of suitable reference functions. In general, a reference function is an arbitrary mapping from a domain of problems into

$\Re^n$. In this paper, we restrict attention to the class G of reference functions defined on Σ_{cc} which satisfy the following:

a. Translation invariance: for all $a \in \Re^n$, $g(a + S) = a + g(S)$.
b. $g(S)$ is a reference point: $g(S) \in comp(S; \nu(S))$ and there exists $x \in S$ such that $x >> g(S)$.
c. Symmetric regularity: If S is symmetric, then $g(S) \in \Delta$
d. Continuity: g is continuous with respect to the Hausdorff topology.

Properties (a) and (c) are straightforward. Since the reference point is used as a basis from which to measure gains, condition (b) requires that it be a suitably chosen element of the feasible set. Property (d) is a technical assumption.

We will use three subclasses of G in the following results. Let us define:

$$G_1 = \{g \in G | g \text{ is scale invariant and } [\mathrm{g(S)} = 0 \Rightarrow \mathrm{g(\bar{S})} = 0]\}$$

$$G_2 = \{g \in G | g \text{ is scale invariant and } [\mathrm{g(S)} = 0 \text{ and } \mathrm{a(S)} \in \Delta \Rightarrow \mathrm{g(\underline{S})} = 0] \text{ if } \underline{\mathrm{S}} \text{ exists}\}$$

$$G_3 = \{g \in G | g \text{ is homogeneous and } [\mathrm{g(S)} = 0 \Rightarrow \mathrm{g(\underline{S})} = 0] \text{ if } \underline{\mathrm{S}} \text{ exists}\}$$

Our proofs require the construction of certain symmetric sets similar to S with the same reference point and the above classes of functions ensure that this is possible.[6] We give examples of plausible reference functions in each class in section 5 below.

Consider the following adaptation of the Nash solution to social choice problems:

$$N^g(S) \equiv \underset{\substack{x \in S \\ x \geq g(S)}}{\operatorname{argmax}} \prod_{i=1}^{n} (x_i - g_i(S)).$$

Our proposal differs from Nash's bargaining solution in that utility gains are measured from the reference point instead of an exogenously specified disagreement point. Similarly, our proposal differs from that of Kaneko and Nakamura [1975] in that they

[6] We thank a referee for comments that lead to these conditions, which considerably simplify the exposition and proofs.

measure utility gains from a pre-specified worst outcome that is a member of every choice set. In our framework, this would correspond to taking the nadir point function, ν, as the reference function.

We show below that this solution may be characterized on the domain Σ_{cc} by axioms analogous to those used by Nash. Notice that in the axiom below, we restrict the application of the independence axiom to those choice sets with the same reference point.

g-Restricted Independence of Irrelevant Alternatives (g-IIA): If $S \subset S'$, $g(S) = g(S')$ and $F(S') \in S$, then $F(S) = F(S')$.

The proof of the next theorem is a straightforward adaptation of the proof of Proposition 1 in Thomson [1981b], which we include for the sake of completeness.

Theorem 4. *For all $g \in G_1$, a social choice rule F satisfies SYM, S.INV, PO and g-IIA on Σ_{cc} if and only if $F = N^g$.*

Proof/

The proof that N^g satisfies the axioms on Σ_{cc} is elementary and is omitted. Conversely, let $S \in \Sigma_{cc}$ be given. By the compactness and convexity of S, and property (b) of the reference function, $N^g(S)$ is well defined and unique. By S.INV we may normalize S such that $g(S) = 0$ and $N^g(S) = \mathbf{1}$. By construction of N^g, S is supported at the point $\mathbf{1}$ by the hyperplane $\sum x_i = n$. Hence $\bar{S}$ is well defined and is supported at the point $\mathbf{1}$ by the hyperplane $\sum x_i = n$. Thus, by SYM and PO, we have that $F(\bar{S}) = \mathbf{1}$. Now because $g \in G_1$, we have that $g(\bar{S}) = 0$. Since $S \subset \bar{S}$, $g(S) = 0 = g(\bar{S})$ and $F(\bar{S}) = \mathbf{1} \in S$, it follows from g-IIA that $F(S) = F(\bar{S}) = \mathbf{1} = N^g(S)$.

■

Remark 4.1. The comprehensiveness assumption is not required for the above result.

Next we introduce the class of the Raiffa choice rules, R^g:

$$R^g(S) \equiv t^* g(S) + (1 - t^*)a(S), \text{ where } t^* = min\{t \in \Re_+ \mid tg(S) + (1-t)a(S) \in S\}.$$

The axioms used are the counterparts of those used by Kalai and Smorodinsky [1975] to characterize the Raiffa bargaining solution on the domain of convex problems with two agents, except that only weak Pareto optimality is used. The generalization to more agents is not immediate since R does not satisfy even Weak Pareto Optimality on Σ_{con} for $n > 2$, see Roth [1980]. This difficulty does not arise on the domain of comprehensive problems, and so we use Σ_{cc} in the treatment that follows. For further discussion see Kalai and Smorodinsky (1975), Thomson (1994) and Conley and Wilkie [1992].

Again we restrict application of the comparison axiom to sets with the same reference point.

g-Restricted Monotonicity (g-R.MON): If $S \subset S'$, $g(S) = g(S')$ and $a(S) = a(S')$, then $F(S') \geq F(S)$.

Theorem 5. *For all $g \in G_2$, a social choice rule F satisfies SYM, S.INV, WPO, CONT and g-R.MON on Σ_{cc} if and only if $F \equiv R^g$*

Proof/

The proof that R^g satisfies the axioms is elementary and is omitted. Conversely let F be a choice rule satisfying the axioms. Assume that $S \in \Sigma_{cc} \cap \Sigma_{s.comp}$. By S.INV the problem can be normalized so that $g(S) = 0$ and $a(S) = (\beta, \ldots, \beta) \equiv y$. Then $R^g(S) = (\alpha, \ldots, \alpha) \equiv x$ for some $\alpha > 0$. For $i = 1, \ldots, n$, let a^i be a vector in $\Re^n$ such that $a^i_i = \beta$ and $a^i_j = 0$ for all $j \neq i$. Let $T = con(0, a^1, a^2, \ldots, a^n, x)$. Since $T \in \Sigma_{cc}$ is a symmetric subset of S, we know that $\underline{S}$ must exist. Since $g(S) = 0$, $a(S) \in \Delta$, and $g \in G_2$, it follows that $g(\underline{S}) = 0$. Since $\underline{S}$ is symmetric, $x \in T$ and $x \in S$, it follows that x is the only symmetric element in $WP(\underline{S})$. Thus, WPO and SYM imply that $F(\underline{S}) = x$. Now $T \subset \underline{S} \subset S$ and $a(T) = a(S)$ by construction. Therefore, $a(S) = a(\underline{S}) = y$ so $g-$R.MON implies that $F(S) \geq F(\underline{S}) = x$. Since $S \in \Sigma_{s.comp}$, we conclude that $F(S) = x$.

To complete the proof, let S be an arbitrary set in Σ_{cc}. and let $\{S^\nu\}$ be a sequence of sets in $\Sigma_{s.comp} \cap \Sigma_{cc}$ such that $S^\nu \to S$. Applying the argument above for each such S^ν, we conclude that $F(S^\nu) = R^g(S^\nu)$. However, g is continuous and R^g is continuous, and so we know that $F(S^\nu) \to R^g(S) = x$. Therefore, by CONT, we conclude that $F(S) = x$.

$\blacksquare$

Remark 5.1. The assumption of convexity is not required for Theorem 5. The proof can be completed following the technique used in Conley and Wilkie [1991].

Remark 5.2. For $g \in G_2$ the choice rule R^g does not satisfy Pareto Optimality when there are more than two agents as shown by the following example:

Let $S \subset \Re^3 = con((0,0,0),(0,1,0),(1,0,0),(0,0,1),(1,0,1),(0,1,1))$, and let $g = \nu$. Then $\nu(S) = (0,0,0)$ and $a(S) = (1,1,1)$, thus $R^\nu(S) = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$, which is Pareto dominated by $(\frac{1}{2},\frac{1}{2},1) \in S$.

We now define a generalized egalitarian solution with respect a to reference function g:

$$E^g(S) \equiv g(S) + \mathbf{1}t^*, \text{ where } t^* = min\{t \in \Re_+ \mid g(S) + \mathbf{1}t \in S\}.$$

We also need to modify the axiom of S.MON to take account of the reference function g.

g-Strong Monotonicity (g-S.MON): If $S \subset S'$ and $g(S) = g(S')$, then $F(S') \geq F(S)$.

Theorem 6. *For all $g \in G_3$, a social choice rule F satisfies WPO, SYM, T.INV, CONT and g-S.MON on Σ_{cc} if and only if $F \equiv E^g$.*

Proof/

The proof that E^g satisfies the five axioms is elementary and is omitted. Conversely let F be a solution satisfying the five axioms. Given $S \in \Sigma_{cc} \cap \Sigma_{s.comp}$, we can assume by T.INV that the problem has been normalized so that $g(S) = 0$. Thus, $E^g(S) = (\alpha, \ldots, \alpha) \equiv x$ for some $\alpha > 0$. Let $T = comp(x; 0)$. Since $T \subset S$, $T \in \Sigma_{cc}$ and T is symmetric, it follows that $\underline{S}$ exists. Since $T \subset \underline{S} \subset S$, $x \in WP(S)$, and $x \in T$, it follows that $x \in WP(\underline{S})$. Thus, by SYM and WPO, $F(\underline{S}) = x$. From the definition of G_3, it follows that $g(\underline{S}) = 0$. Since $g(S) = g(\underline{S})$ and $\underline{S} \subset S$, it follows from g-S.MON that $F(S) \geq x$. Since $x \in P(S)$, we can apply WPO to conclude that $F(S) = x = E^g(S)$.

To extend the theorem to an arbitrary set in Σ_{cc}, take a sequence of sets $\{S^\nu\}$ in $\Sigma_{s.comp} \cap \Sigma_{cc}$ such that $S^\nu \to S$. Applying the above argument to each such S^ν, we conclude that $F(S^\nu) = E^g(S^\nu)$. The continuity of g implies that E^g is also continuous so that $F(S^\nu) \to E^g(S) = x$. Therefore by CONT it follows that $F(S) = x = E^g(S)$.

■

Remark 6.1. As in the case of Theorem 5, the assumption of convexity is not required for Theorem 6. Again, the proof can be completed following the technique used in Conley and Wilkie [1991].

5. Discussion

In section 4, we defined the classes of reference functions G_1, G_2 and G_3, that are "compatible" with the Nash, Raiffa and egalitarian solutions, respectively. Each of the classes is defined by properties that are satisfied by the nadir point mapping. In particular, $\nu(\cdot) \in G_1 \cap G_2 \cap G_3$, see Corollary 1 below. For the classes G_i to be more interesting however, we must show that each class contains more than the nadir point mapping. In the results that follow we provide examples of reference functions in each class different from the nadir point mapping. Each of these may be though of as a "compromise" between the nadir point map and some other reasonable reference point.

To begin we need a simple lemma that provides the relationship between the nadir point and the ideal point of the sets S, $\bar{S}$, and $\underline{S}$.

Lemma 1. *Let $S \in \Sigma_{cc}$. Then $\nu(\bar{S}) = (min_{i \in N}\{\nu_i(S)\})\mathbf{1}$ and $a(\bar{S}) = (max_{i \in N}\{a_i(S)\})\mathbf{1}$. If $\underline{S}$ exists, then $\nu(\underline{S}) = (max_{i \in N}\{\nu_i(S)\})\mathbf{1}$ and $a(\underline{S}) = (min_{i \in N}\{a_i(S)\})\mathbf{1}$.*

Proof/

Recall that $\bar{S} = concomp\{\cup_{\pi \in \Pi^n} \pi(S)\}$, and $\underline{S} = \cap_{\pi \in \Pi^n} \pi(S)$ when $\underline{S}$ exists. Let $\bar{x} = min_{i \in N}\{\nu_i(S)\}\mathbf{1}$. Clearly $\bar{x} \leq z$ for all $z \in \bar{S}$. Furthermore, there exists for each i a point $z^i \in \cup_{\pi \in \Pi^n} \pi(S)$ such that $(z^i)_i = min_{i \in N}\{\nu_i(S)\}$. Hence $\nu_i(\bar{S}) = min_{i \in N}\{\nu_i(S)\}$ for each i, and $\bar{x} = \nu(\bar{S})$. The other arguments are similar.

■

We now turn to the examples.

Theorem 7. *If $\alpha \in [0, \frac{1}{n}[$ and $g : \Sigma_{cc} \rightarrow \Re^n$ is given by $g(S) = (1-\alpha)\nu(S) + \alpha a(S)$, then $g \in G_1$.*

Proof/

It is straightforward to show that conditions (a), (c), and (d) in the definition of a reference function are satisfied. To show that (b) is satisfied, note that for each i, $(\nu_{-i}(S), a_i(S)) \in S$ where $(\nu_{-i}(S), a_i(S))$ is the point $\nu(S)$ with $a_i(S)$ replacing $\nu_i(S)$. Convexity implies that

$$\sum_{i \in N} 1/n(\nu_{-i}(S), a_i(S)) = \nu(S) + 1/n[a(S) - \nu(S)] \in S.$$

Since $0 \leq \alpha < 1/n$ and $a(S) >> \nu(S)$ it follows that

$$\nu(S) + 1/n[a(S) - \nu(S)] >> g(S) \geq \nu(S).$$

Hence condition (b) is satisfied, and $g \in G$.

To show that $g \in G_1$, note that both $a(\cdot)$ and $\nu(\cdot)$ are scale invariant so g is scale invariant. Now choose S such that $g(S) = 0$. It remains to show that $g(\bar{S}) = 0$. From Lemma 1, it follows that $\nu(\bar{S}) = (min_{i \in N}\{\nu_i(S)\})\mathbf{1}$ and $a(\bar{S}) = (max_{i \in N}\{a_i(S)\})\mathbf{1}$. Choose j such that $a_j(S) = max_{i \in N}\{a_i(S)\}$. Since $g_j(S) = (1-\alpha)n_j(S) + \alpha a_j(S) = 0$, it follows that, $\nu_j(S) = min_{i \in N}\{\nu_i(S)\}$. Therefore,

$$0 = (1-\alpha)\nu_j(S)\mathbf{1} + \alpha a_j(S)\mathbf{1} = (1-\alpha)\nu(\bar{S}) + \alpha a(\bar{S}) = g(\bar{S})$$

and the proof is complete.

■

Theorem 8. *If $\alpha \in [0,1[$ and $g : \Sigma_{cc} \to \Re^n$ is given by $g(S) = (1-\alpha)\nu(S) + \alpha R^{\nu}(S)$, then $g \in G_2$.*

<u>Proof/</u>

It is straightforward to show that $g \in G$. To show that $g \in G_2$, note that scale invariance follows from the scale invariance of $a(\cdot)$ and $\nu(\cdot)$. Now choose S such that $g(S) = 0$ and $a(S) \in \Delta$, and suppose that $\underline{S}$ exists. It remains to show that $g(\underline{S}) = 0$. First, note that there exists $\lambda \in]0,1[$ such that $R^{\nu}(S) = \lambda\nu(S) + (1-\lambda)a(S)$. Since $g(S) = 0$ it follows that $[(1-\alpha)+\alpha\lambda]\nu(S) = -\alpha(1-\lambda)a(S)$. Therefore, as $a(S) \in \Delta$ we conclude that $\nu(S) \in \Delta$ and $R^{\nu}(S) \in \Delta$. Applying Lemma 1, it follows that $\nu(\underline{S}) = max_{i\in N}\{\nu_i(S)\})\mathbf{1} = \nu(S)$ and $a(\underline{S}) = min_{i\in N}\{a_i(S)\})\mathbf{1} = a(S)$, so that $\{R^{\nu}(\underline{S})\} = \Delta \cap WP(\underline{S})$. To complete the proof note that $R^{\nu}(S) \in WP(\underline{S})$. (If not, then there exists $x \in \underline{S}$ such that $x >> R^{\nu}(S)$. Since $x \in S$ and $R^{\nu}(S) \in WP(S)$ we have a contradiction.) Hence, $R^{\nu}(S) \in \Delta \cap WP(\underline{S}) = \{R^{\nu}(\underline{S})\}$ from which it follows that $R^{\nu}(S) = R^{\nu}(\underline{S})$. Since $\nu(S) = \nu(\underline{S})$ we conclude that $g(S) = g(\underline{S}) = 0$ and the proof is complete.

■

Theorem 9. *If $\alpha \in [0,\frac{1}{n}[$ and $g : \Sigma_{cc} \to \Re^n$ is given by $g(S) = (1-\alpha)\nu(S) + \alpha(min_{i\in N}\{a_i(S)\})\mathbf{1}$, then $g \in G_3$.*

<u>Proof/</u>

Using the arguments of Theorem 7, it is straightforward to show that $g \in G$. To show that $g \in G_3$, note that the homogeneity of g follows from the homogeneity of $\nu(\cdot)$ and $a(\cdot)$. Now choose $S \in \Sigma_{cc}$ with $g(S) = 0$ and suppose that $\underline{S}$ exists. Since $g(S) = 0$, it follows that $\nu(S) \in \Delta$, so Lemma 1 implies that $\nu(S) = \nu(\underline{S})$ and $a(\underline{S}) = min_{i\in N}\{a_i(S)\})\mathbf{1}$. Therefore $g(\underline{S}) = g(S) = 0$, and the proof is complete.

■

The following result is an immediate consequence of the previous three theorems.

Corollary 1. *If $g : \Sigma_{cc} \to \Re^n$ is the nadir point mapping, $\nu(\cdot)$, then $g \in G_1 \cap G_2 \cap G_3$.*

6. Conclusion

We make two points in this paper. First, we show several impossibility theorems for social choice rules. We argue that the reason for these negative results is that the comparison axioms are too strong when we have no means of measuring the relative gains or losses of agents. Second, we suggest that Thomson's [1982b] idea of defining social choice rules with reference points is one way to recover possibility theorems. We discuss this idea at greater length in Conley, McLean, and Wilkie [1994]. Thus, it is not the existence of the disagreement point that make characterizations of solution concepts possible in bargaining theory and impossible in social choice theory. Rather, the disagreement point is a special case of the more general idea of a reference point from which the relative utility gains and losses of agents may be measured.

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